MATH 267 Fall 2026 - Class Notes

11/4/2026

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Summary: In class we introduced sequences of real numbers as functions $\mathbb{N} \to \mathbb{R}$, along with bounds, infimum, supremum, and increasing/decreasing sequences, and worked a full example with $x_n = \frac{n}{n+3}$. We finished with the recursion theorem and the difference between recursive definitions and closed forms.

Notes:

Review (proof from quiz). Let $f : A \to B$ be onto and $g : B \to C$ be onto. *Goal:* show $g \circ f : A \to C$ is onto.

Proof. Let $c \in C$. Since g is onto, there exists $b \in B$ such that $g(b) = c$. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Now $(g \circ f)(a) = g(f(a)) = g(b) = c$. Since c was arbitrary, $g \circ f$ is onto. □

Warm-up. Consider the sequence $x_n = \dfrac{2n-1}{n+2}$.

$$x_0 = -\tfrac{1}{2}, \qquad x_1 = \tfrac{1}{3}, \qquad x_2 = \tfrac{3}{4}, \qquad x_3 = 1.$$

- **Infimum** (greatest lower bound): $-\frac{1}{2}$ is the infimum.
- **Supremum** (least upper bound): conjecture, possibly 2.

Prove 2 *is an upper bound:*

$$\frac{2n-1}{n+2} < \frac{2n}{n+2} < \frac{2n}{n} = 2$$

(the second inequality is for $n \geq 1$; for $n = 0$ we just have $x_0 = -\frac{1}{2} < 2$). So 2 is an upper bound. In this case the supremum is 2. Proving it would require showing that no smaller number is also an upper bound.

Sequences.

- A sequence x_n of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ where $x_n = f(n)$.
- Alternatively, every sequence is also a set of real numbers $\{x_n : n \in \mathbb{N}\}$.
  - The infimum and supremum of a sequence are exactly the same as those of the corresponding set of real numbers.

Worked example. Let $x_n = \dfrac{n}{n+3}$. Prove that x_n is bounded, is strictly increasing, and find and prove its infimum and supremum.

- **Bounded above.** Find $M \geq \frac{n}{n+3}$. Since $n < n+3$, dividing both sides by $n+3$ gives
$$\frac{n}{n+3} < \frac{n+3}{n+3} = 1,$$
so 1 is an upper bound.

- **Bounded below.** Since $n \geq 0$ for all n and $n+3 > 0$ for all n, $\frac{n}{n+3} \geq 0$, so 0 is a lower bound for the sequence.

- Since it is bounded above and below, it is bounded.

- **Strictly increasing.** To show that x_n is strictly increasing, compute $x_{n+1} - x_n$:
$$\begin{aligned} x_{n+1} - x_n &= \frac{n+1}{(n+1)+3} - \frac{n}{n+3} = \frac{n+1}{n+4} - \frac{n}{n+3} \\ &= \frac{(n+1)(n+3)}{(n+4)(n+3)} - \frac{n(n+4)}{(n+3)(n+4)} \\ &= \frac{(n+1)(n+3) - n(n+4)}{(n+4)(n+3)} \\ &= \frac{(n^2+4n+3) - (n^2+4n)}{(n+4)(n+3)} \\ &= \frac{3}{(n+4)(n+3)}. \end{aligned}$$
The numerator and denominator are positive, so $x_{n+1} - x_n > 0$, thus $x_{n+1} > x_n$.

- **Infimum.** Since the sequence is increasing, $\inf(x_n) = x_0 = \frac{0}{3} = 0$.

- **Supremum.** *Claim:* 1 is the supremum of $\{x_n\}$. We already know 1 is an upper bound for $\{x_n\}$. It remains to show no smaller number is an upper bound.

  Suppose that $p < 1$ is also an upper bound for our sequence. Write
$$x_n = \frac{n}{n+3} = 1 - \frac{3}{n+3}.$$
Since $p < 1$, $1 - p > 0$. We want $1 - \frac{3}{n+3} > p$, i.e. we want $1 - p > \frac{3}{n+3}$:
$$n + 3 > \frac{3}{1-p} \qquad \Longleftrightarrow \qquad n > \frac{3}{1-p} - 3.$$
By the Archimedean principle it is always possible to pick $n > \frac{3}{1-p} - 3$. Pick such an n. Then (replacing n by something smaller)
$$x_n = 1 - \frac{3}{n+3} > 1 - \frac{3}{\left(\frac{3}{1-p} - 3\right) + 3} = 1 - \frac{3}{\frac{3}{1-p}} = 1 - (1-p) = p.$$
Since this $x_n > p$, p is <u>not</u> an upper bound for x_n. So 1 is the least upper bound for x_n and is the supremum.

Recursion.

- Since $g : \mathbb{N} \to X$, it gives a corresponding sequence $x_n = g(n)$. This means we can work with recursively defined sequences and the result should always be well-defined.
- **Example:** $x_0 = 3$, $x_{n+1} = 2x_n + 1$ for all $n \geq 0$ gives a unique sequence (only one sequence satisfies this recursion).
- **Example:** $a_1 = 1$, $a_{n+1} = 2a_n + 1$ (recursive form). Then $a_2 = 3$, $a_3 = 7$, $a_4 = 15$, and $a_n = 2^n - 1$ (closed form).
- Usually prefer closed forms to recursive definitions. Often use induction to prove that a recursive definition and closed form give the same definition.

Definitions:

Definition 1. *A **sequence** x_n of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ where $x_n = f(n)$.*

Definition 2. *A sequence is **bounded above** if there exists M with $M \geq x_n$ for all n.*

Definition 3. *A sequence is **bounded below** if there exists m with $m \leq x_n$ for all n.*

Definition 4. *A sequence is **bounded** if it is bounded above and below.*

Definition 5. *The **infimum** exists iff it is bounded below. The **supremum** exists iff it is bounded above.*

Definition 6. *A sequence is **increasing** if $x_n \leq x_{n+1}$ for all n. It is **strictly increasing** if $x_n < x_{n+1}$ for all n.*

Definition 7. *A sequence is **decreasing** if $x_n \geq x_{n+1}$ for all n. It is **strictly decreasing** if $x_n > x_{n+1}$ for all n.*

Definition 8. *A formula for a sequence where x_{n+1} depends on x_n (and maybe previous terms) is called a **recursive definition** for a sequence.*

Definition 9. *If we can write $x_n = f(n)$, where $f(n)$ does not involve sequence terms, this is a **closed form** for our sequence.*

Theorems and Proofs:

Theorem 1. *If x_n is increasing, then the first term is the infimum (also the minimum).*

Theorem 2 (Recursion Theorem)**.** *If $f : X \to X$ is any function and $a \in X$, then there is a unique function $g : \mathbb{N} \to X$ such that $g(0) = a$ and $g(n + 1) = f(g(n))$ for all $n \geq 0$.*

In general, $g(n) = f^{(n)}(a)$, i.e. f composed with itself n times. For example, $g(1) = f(a)$ and $g(2) = f(g(1)) = f(f(a))$.

Questions raised in class: None were recorded in the handwritten notes.