MATH 267 Fall 2026 - Class Notes

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Summary: Today we covered images and inverse images (or preimages). We went over how they're defined and how we prove statements that use them.

Definitions:
Let f be a function where $f : X \to Y$ and $A \subseteq X$, $B \subseteq Y$, $x \in X$, $y \in Y$

The image of A under f is then defined as $f(A) = \{f(a) : a \in A\}$. Note that this is a set and is different from $f(x)$ where $f(x)$ is the value of f at x and is not a set, as opposed to $f(\{x\})$ which is the image of $\{x\}$ and is a set.

Then we have $f^{-1}(B) = \{x \in X : f(x) \in B\}$ which is called the inverse image of B under f(also called the preimage).

$^*f^{-1}(B)$ is defined for any function f even if f^{-1} doesn't exist as a function, you don't need to have a bijection in order to have an inverse image.

Notes:

- The warm up we did today had a good discrete example of images.

t	1	2	3	4	5
f(t)	a	b	a	c	a

  With $A = \{1, 2, 4\}$ and $B = \{a, c\}$, this example is simple and you can see $f(A) = \{a, b, c\}$ and $f^{-1}(B) = \{1, 3, 4, 5\}$

- Then the first example in class we were given $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, notice this function is not a bijection, it is neither one-to-one or onto as $f(-2) = f(2)$ and nothing maps to -1. With this function we can find the image of an interval, here we did the image of the interval $[-1, 1]$, where $f([-1, 1]) = [0, 1]$. This example brought up an important question to note as well. One student asked about the interval being a set, as the interval $[0, 1]$ is just another way to write a set as $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, as compared to $f(\{-1, 1\}) = \{1\}$ which is not an interval.
  Continuing with this example we found a preimage $f^{-1}((4, 9]) = (2, 3] \cup [-3, -2]$. Notice our preimage took 1 interval and made 2 intervals even though $f^{-1}(x)$ doesn't exist since f isn't a bijection. Although don't forget $f(x)$ could be a bijection if we restricted the domain from for all $x \in \mathbb{R}$ to just the positive real numbers.

- Also if $f : X \to Y$ then the image of the domain is the range of f or $f(X) = ran(f)$

Theorems and Proofs:

Example 1. *Let* $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^2 + 1$, *Find and prove* $f^{-1}([3, 9])$
First we use what we learned from the first example and conjecture that $f^{-1}([3, 9]) = [-2, 1] \cup [1, 2]$.
Now were proving equality so we must prove it using double inclusion.
Proof: First we show the first inclusion $f^{-1}([3, 9]) \subseteq [-2, 1] \cup [1, 2]$, *let* $x \in f^{-1}([3, 9])$ *so* $f(x) \in [3, 9]$ *meaning* $2x^2 + 1 \in [3, 9]$, *so then* $3 \leq 2x^2 + 1 \leq 9$, *then equivalently by subtracting* 1, $2 \leq 2x^2 \leq 8$, *then dividing by* 2, $1 \leq x^2 \leq 4$, *so this means either* $-2 \leq x \leq -1$ *or* $1 \leq x \leq 2$. *So in either case* $x \in [-2, -1] \cup [1, 2]$, *so check on the first inclusion.*
Then the second inclusion $[-2, 1] \cup [1, 2] \subseteq f^{-1}([3, 9])$, *let* $x \in [-2, -1] \cup [1, 2]$, *then* $-2 \leq x \leq -1$ *or* $1 \leq x \leq 2$, *so if* $-2 \leq x \leq -1$ *then squaring gives* $1 \leq x^2 \leq 4$, *so* $2 \leq 2x^2 \leq 8$, *and adding* 1, $3 \leq 2x^2 + 1 \leq 9$, *then* $3 \leq f(x) \leq 9$, *so* $x \in f^{-1}([3, 9])$ *by the same argument if* $1 \leq x \leq 2$ *then* $x \in f^{-1}([3, 9])$, *thus the second inclusion holds so therefore by double inclusion* $f^{-1}([3, 9]) = [-2, 1] \cup [1, 2]$. $\square$

Example 2. *Theorem: Suppose* $f : X \to Y$ *and* $A_1, A_2 \subseteq X$, *then* $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
Again were proving equality so we use double inclusion.
proof: First inclusion $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$, *let* $y \in f(A_1 \cup A_2)$, *then* $\exists x \in (A_1 \cup A_2)$ *such that* $f(x) = y$, *this means* $x \in A_1$ *or* $x \in A_2$. *Suppose* $x \in A_1$, *then* $y \in f(A_1)$ *since* $f(x) = y$. *Then by the same argument if* $x \in A_2$ *then* $y \in f(A_2)$ *where* $f(A_1) \subseteq f(A_1) \cup f(A_2)$ *and* $f(A_2) \subseteq f(A_1) \cup f(A_2)$. *In either case* $y \in f(A_1) \cup f(A_2)$ *so* $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$.
Now for the second inclusion $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$, *let* $y \in f(A_1) \cup f(A_2)$, *then* $y \in f(A_1)$ *or* $y \in f(A_2)$, *then if* $y \in f(A_1)$ *then* $\exists x \in A_1$ *with* $f(x) = y$. *Since* $x \in A_1 \subseteq A_1 \cup A_2$ *then* $y \in f(A_1 \cup A_2)$ *and the same argument if* $y \in f(A_2)$. *In either case* $y \in A_1 \cup A_2$ *so* $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$. *Thus* $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. $\square$

Example 3. *Suppose* $f : X \to Y$ *and* $B_1, B_2 \subseteq Y$ *with* $B_1 \subseteq B_2$ *then* $f^{-1}(B_1) \subseteq f^{-1}(B_2)$
Proof: Let $x \in f^{-1}(B_1)$ *then* $f(x) \in B_1 \subseteq B_2$ *so* $f(x) \in B_2$, *therefore* $x \in f^{-1}(B_2)$, *thus* $f^{-1}(B_1) \subseteq f^{-1}(B_2)$. $\square$