MATH 267 Fall 2026 - Class Notes

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Summary: Images and inverse images of functions. Instead of using a singular element as an input for a function, using a set as the input with result in the image of a function which is a set containing each output pertaining to the corresponding elements within the input set.

Notes:

Example: $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$. Not a bijection.
$f([-1, 1]) = [0, 1]$
Note: $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$
$f(\{-1, 1\}) = \{1\}$
$f^{-1}((4, 9]) = (2, 3] \cup [-3, -2)$ (even though $f^{-1}(x)$ doesn't exists since f inst a bijection)

Example: $g : \mathbb{Q} \longrightarrow \mathbb{Q}, g(x) = x^2$
$g(\frac{2}{3}) = \frac{4}{9}$
$g(\{-1, 1\}) = \{1\}$
$g^{-1}(\{5\}) = \emptyset$

How to prove: The main method to prove the image of a certain function is through double inclusion.

Example of proof: Let $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 2x^2 + 1$ The goal is to find and prove that $f^{-1}([3, 9])$. Conjecture is

$$f^{-1}([3, 9]) = [-2, -1] \cup [1, 2]$$

Proof: Let $x \in f^{-1}([3, 9])$. So $f(x) \in [3, 9]$, meaning that $2x^2+1 \in [3, 9]$. So $3 \leq 2x^2+1 \leq 9$. Equivalently, $2 \leq 2x^2 \leq 8$ and $1 \leq x^2 \leq 4$. This means that either $-2 \leq x \leq -1$ or $1 \leq x \leq 2$. So $x \in [-2, -1] \cup [1, 2]$. Thus $f^{-1}([3, 9]) \subseteq [-2, -1] \cup [1, 2]$.

Now, let $x \in [-2, -1] \cup [1, 2]$. Then either $-2 \leq x \leq -1$ or $1 \leq x \leq 2$. If $-2 \leq x \leq -1$, then squaring both sides we find that $1 \leq x^2 \leq 4$. So $2 \leq 2x^2 \leq 8$ and $3 \leq 2x^2 + 1 \leq 9$. Thus $3 \leq f(x) \leq 9$. So $x \in f^{-1}([3, 9])$; by the same argument, if $1 \leq x \leq 2$, then $x \in f^{-1}([3, 9])$. Therefore $[-2, -1] \cup [1, 2] \subseteq f^{-1}([3, 9])$. Finally, by proof of double inclusion, $f^{-1}([3, 9]) = [-2, -1] \cup [1, 2]$. $\square$

Definitions:

Definition 1.

$$F : X \longrightarrow Y$$

$$A \subseteq X, B \subseteq Y$$

$$x \in X, y \in Y$$

$$F(A) = \{F(a) : a \in A\}$$

$F(A)$ in this case is called the image of A under F.

$F(x)$ = value of x at x (not a set)

$F(\{x\})$ = image of $\{x\}$ (which is a set).

$F^{-1}(B) = \{x \in X : f(x) \in B\}$ is the inverse image of B under f (also called the pre-image). The inverse image is defined for any function f even if f^{-1} doesn't exist as a function.

Note that the image of the domain of a function is its range.

Theorems and Proofs:

Theorem 1. *Suppose $F : X \longrightarrow Y$ and $A_1, A_2 \subseteq X$, then*

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

Proof. Let $y \in f(A_1 \cup A_2)$, then $\exists x \in A_1 \cup A_2$ with $f(x) = y$. This means that $x \in A_1$ or $x \in A_2$. Suppose $x \in A_1$, then $y \in f(A_1) \subseteq f(A_1) \cup f(A_2)$ since $f(x) = y$. If $x \in A_1$, then $y \in f(A_2) \subseteq f(A_1) \cup f(A_2)$. In either case, $y \in f(A_1) \cup f(A_2)$, so $f(A_1 \cup A_2) \leq f(A_1) \cup f(A_2)$. Thus $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$.

Now, let $y \in f(A_1)$ or $y \in f(A_2)$. If $y \in f(A_1)$, then $\exists x \in A_1$ with $f(x) = y$. Since $x \in A_1 \subseteq A_1 \cup A - 2$, then $y \in f(A_1 \cup A_2)$. The same argument applies if $y \in f(A - 1)$. In either case, $y \in f(A_1 \cup A_2)$ so $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$. Therefore, $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$. □