MATH 267 Fall 2026 - Class Notes

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Summary: Today, we continued talking about limits and convergence of sequences. We went over a warm up involving a recursive series, and proved a few theorems related to convergence.

Notes:

Warm up:

1. $x_n = \frac{2n-1}{5n-1} \to \frac{2}{5}$ and $|x_n - \frac{5}{2}| = \frac{7}{5(5n-1)}$

How large should N be so that $n > N \to \frac{7}{5(5n-1)} < \epsilon$

$\frac{7}{5(5n-1)} < \frac{7}{5(5N-1)} < \epsilon$ We can solve for N.

$\frac{7}{5(5N-1)} < \epsilon$ $\frac{1}{5N-1} < \frac{5\epsilon}{7}$ $5N - 1 > \frac{7}{5\epsilon}$ $5N > \frac{7}{5\epsilon} + 1$ $N > \frac{7}{25\epsilon} + \frac{1}{5}$. This does work, but we can make an easier fraction.

When $N > 1$, $\frac{7}{5(5N-1)} < \frac{7}{5(5N-N)} = \frac{7}{20N} < \epsilon, N > \frac{7}{20\epsilon}$. This is easier to use in a proof.

2. Suppose $x_1 = 2, x_{n+1} = \frac{1}{4}x_n + 3$. If we know that $x_n \to L$ for some L, what is L?

If $x_n \to L$, then $x_{n+1} \to L$.

So, $L = \frac{1}{4}L + 3$. Now solve for L. L=4

This only works if you already know the sequence converges

- If $a < b$, then $\frac{1}{a} > \frac{1}{b}$. If $a > b$, then $-a < -b$
- A divergent sequence does not mean an unbounded one

Theorems and Proofs:

Theorem 1. *If a sequence x_n converges to L, then the limit is unique.*

Theorem 2. ***Triangle Inequality:*** $|x + y| \leq |x| + |y|$

Prove the limit of a sequence is unique using the triangle inequality

Proof. Suppose for contradiction, there exists a sequence x_n that converges to L and L' where $L \neq L'$. Since $L \neq L, |L - L'| = d > 0$. Pick $\epsilon = \frac{d}{3} > 0$. Since $x_n \to L$ there exists N so that $n > N$ means $|x_n - L| < \epsilon$. And, since $x_n \to L'$ there exists N' so that $n > N'$ means $|x_n - L'| < \epsilon$. Now pick $n > Max(N, N')$. So $|x_n - L| < \epsilon$. Now consider $d = |L - L'| = |(L - x_n) + (x_n - L')| \leq |L - x_n| + |x_n - L'| = |x_n - L| + |x_n - L'| < \epsilon + \epsilon = \frac{2d}{3}$. This is a contradiction, so there must be only one number that the sequence converges to. □

Theorem 3. *If x_n converges, then it is bounded. If x_n is unbounded, then it diverges*

Theorem 4. ***Algebraic Limit Theorem:*** *If x_n and y_n are two sequences that converge to L and M respectively and $\alpha \in \mathcal{R}$. Then:*

- $\alpha x_n \to \alpha L$

  Proof. Suppose $x_n \to L$ and $\alpha \in \mathcal{R}$. Fix $\epsilon > 0$. Since $x_n \to L$, then for any ϵ' there exists an N' so that $n > N'$ means $|x_n - L| < \epsilon'$. Pick $\epsilon = \frac{\epsilon}{|\alpha|+1}$. Then there exists an N' so that $n > N'$ means $|x_n - L| < \epsilon' = \frac{\epsilon}{|\alpha|+1}$. Let N=N'. If $n > N$ then $|\alpha x_n - \alpha L| = |\alpha(x_n - L)| = |\alpha||x_n - L| < |\alpha|(\frac{\epsilon}{|\alpha|+1}) = \frac{|\alpha|}{|\alpha|+1}(\epsilon) < \epsilon$. □

- $x_n + y_n \to L + M$

- $x_n y_n \to LM$

  Proof. Suppose $x_n \to L$ and $y_n \to M$. Since x_n converges, it is bounded so there exists a $k > 0$ so that $|x_n| < k$ for all n. Since $x_n \to L$ and $y_n \to M$, we can pick N_1 and N_2 so that $|x_n - L| < \epsilon_1$ and $|y_n - M| < \epsilon_2$. Fix $\epsilon > 0$, set $\epsilon_2 = \frac{epsilon}{2k}$. Consider $|x_n y_n - LM| = |x_n y_n - x_n M + x_n M - LM| = |x_n(y_n - M) + (x_n - l)M| \leq |x_n(y_n - M)| + |(x_n - L)M|$ (triangle inequality). $= |x_n||y_n - M| + |x_n - L||M| = |x_n||y_n - M| + |x_n - L||M| \leq |x_n|(\frac{\epsilon}{2k}) + \frac{\epsilon}{2|\alpha|+1}|M| \leq k(\frac{\epsilon}{2k}) + \epsilon(\frac{|M|}{2|\alpha|+1}) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $x_n y_n \to LM$ □

- *if* $L \neq 0$, *then* $\frac{1}{x_n} \to \frac{1}{L}$