

# MATH 267 Fall 2026 - Class Notes

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Scribe: Zoë Barnes

**Summary:** Today we began class with a warm up on sets, subsets, and power sets. We discussed when something is an element or a subset of a set. We continued class with proofs by induction and strong induction.

## Notes:

- Only a set can be a subset of another set, so just the element 2 could not be a subset of any set, but  $\{2\}$  could be.
- The empty set  $\emptyset$  is a subset of every set.

## Definitions:

**Definition 1.** The **Power Set**  $\mathcal{P}(S)$  is the set of all sets  $X$  that satisfy  $X \in \mathcal{P}(S) \iff X \subseteq S$ . In words it is the set of all possible subsets of  $S$ .

**Definition 2.** **Cardinality**, denoted  $|S|$ , is the number of elements in a set.

**Definition 3.** The **element method** is a way to show one set is a subset of another. To use it, let  $x$  be an arbitrary element of  $S$  and show it's also an element of  $T$ .

## Theorems and Proofs:

**Theorem 1.** If  $|S| = n$  then  $|\mathcal{P}(S)| = 2^n$ .

*Proof by induction on  $n$ :*

*Base case:  $n = 0$*

*If  $|S| = 0$  then  $S = \emptyset$  and  $\mathcal{P}(S) = \mathcal{P}(\emptyset) = \{\emptyset\}$*

*so  $|\mathcal{P}(S)| = 1 = 2^0$ .*

*Induction Step: Suppose for every set  $S$  with  $|S| = n$ ,  $|\mathcal{P}(S)| = 2^n$ .*

*Suppose  $T$  is a set with  $n + 1$  elements.*

*Since  $n + 1 > 0$ ,  $T$  has at least one element,  $t$ .*

*Then  $T' = T \setminus \{t\}$  has  $n$  elements.*

*By our induction hypothesis*

*$\mathcal{P}(T')$  has  $2^n$  elements.*

*Now let  $U \subseteq T$ .*

*Consider two cases.*

*Case 1:  $t \notin U$ .*

*Every element of  $U$  is an element of  $T$  not equal to  $t$ , so every element of  $U$  is in  $T'$ , so  $U \subseteq T'$ .*

*So  $U \in \mathcal{P}(T')$ .*

By our induction hypothesis there are  $2^n$  possibilities for  $U$ .

Case 2:  $t \in U$ .

Let  $U' = U \setminus \{t\}$ .

Now  $U' \subseteq T'$

So  $U' \in \mathcal{P}(T')$

There are  $2^n$  possibilities for  $U'$ . Each of those gives a unique  $U = U' \cup \{t\}$   $2^n$  possible sets  $U$ .

So total number of elements of  $\mathcal{P}(T)$  is  $2^n + 2^n = 2^{n+1}$ .  $\square$

**Theorem 2.** Let  $A_1, A_2, \dots, A_N$  be any collection of sets. Then  $\bigcap_{i=1}^N \mathcal{P}(A_i) = \mathcal{P}(\bigcap_{i=1}^N A_i)$

Proof using strong induction on  $N$ :

Base case:  $N = 1$ .

$\bigcap_{i=1}^1 \mathcal{P}(A_i) = \mathcal{P}(\bigcap_{i=1}^1 A_i)$ .

This base case follows from  $\bigcap_{i=1}^1 A_i = A_1$ .

Induction step: Suppose for any  $1 \leq j \leq N$  and for any sets  $A_1, A_2, \dots, A_j$  that  $\bigcap_{i=1}^j \mathcal{P}(A_i) = \mathcal{P}(\bigcap_{i=1}^j A_i)$ .  
 $\mathcal{P}(A_1) \cap \mathcal{P}(A_2) \dots \cap \mathcal{P}(A_{N+1}) = (\bigcap_{i=1}^N \mathcal{P}(A_i)) \cap \mathcal{P}(A_{N+1}) = \mathcal{P}(\bigcap_{i=1}^N A_i) \cap \mathcal{P}(A_{N+1})$  by our induction hypothesis for  $N$ .

$= \mathcal{P}(A_1) \cap \mathcal{P}(A_2) = \mathcal{P}(A_1 \cap A_2)$  by the  $N = 2$  version of our induction hypothesis.

$= \mathcal{P}((\bigcap_{i=1}^N A_i) \cap A_{N+1}) = \mathcal{P}(\bigcap_{i=1}^{N+1} A_i)$ .  $\square$

Because the proof requires  $N = 2$  we also need that to be a base case.

**Questions raised in class:** One question asked was does a subset have to have brackets? We found that yes, in order to be a subset, the object must have brackets and be a set. Another question raised in class was is  $\emptyset \in S$ ? We determined that no, the empty set is not an element of a set unless it is directly stated that it is one of the objects in  $S$ .