MATH 267 Fall 2026 - Class Notes

10/28/2026

Jenna Klein

Summary: Inverses and Composition - we began class with a warm-up covering the topics of functions from the last two chapters.

Warm-Up: Today's warm-up covered onto and one-to-one functions, composites, and inverse functions.

1. $f : \{1, 2, 3, 4\} \to \{a, b, c\}$ and $g : \{a, b, c\} \to \{x, y\}$

t	1	2	3	4
f(t)	b	a	b	a

s	a	b	c
g(s)	y	x	x

- The table for $g \circ f(t) = g(f(t))$ was not filled out, but we found the following values.

t	1	2	3	4
$g \circ f(t)$	x	y	x	y

- We found that $f(x)$ is not one-to-one since $f(1) = f(3) = b$. We also said that it is not onto because $f(x)$ is never equal to c.
- We also found that $g(s)$ is not one-to-one because $g(b) = g(c) = x$. However, it is onto since all of the elements of the codomain are mapped to.
- The composition $g \circ f(t) = g(f(t))$ is not one-to-one but is onto.

Notes:

How can we show that a function is the inverse of another function? Specifically, how can we show that the function $g(x) : B \to A$ is the inverse of $f(x) : A \to B$?

1. From the definition: $g(y) = x$ iff $f(x) = y$ for every $x \in A$ and $y \in B$.
2. If we already know that $f(x)$ is a bijection, then it suffices to show either $f \circ g(x) = i_B$ or $g \circ f(x) = i_A$ where i_A is the identity function $i_A = A \to A$ and $i_A(x) = x$
3. If we don't already know that $f(x)$ is a bijection, we can show that $f \circ g(x) = i_B$ and $g \circ f(x) = i_A$. This shows us that both of the functions are bijections.

Last Time: We showed that $h(n) : \mathbb{Z} \to \mathbb{Z}$ given by:

$$h(n) = \begin{cases} \text{n+1} & \text{n is odd} \\ \text{n-3} & \text{n is even} \end{cases}$$

is a bijection. We want to find the inverse function $h^{-1} : \mathbb{Z} \to \mathbb{Z}$ and prove it is the inverse. We proposed that $h^{-1}(n) : \mathbb{Z} \to \mathbb{Z}$ is

$$h^{-1}(n) = \begin{cases} \text{n+3} & \text{n is odd} \\ \text{n-1} & \text{n is even} \end{cases}$$

Proof. Since we already know that $h(n)$ is a bijection, we can use our second method to prove our proposed inverse function. We will show that $h \circ h^{-1} = i_{\mathbb{Z}}$ or $h^{-1} \circ h = i_{\mathbb{Z}}$. We will show that $h \circ h^{-1} = i_{\mathbb{Z}}$. We will compute the values of this function depending on the parity of n.
Case 2: n is even. $h \circ h^{-1} = h(h^{-1}(n)) = h(n-1)$. Since n is even, then $n-1$ is odd. So $h(n-1) = (n-1)+1 = n$.
Case 2: n is odd. $h \circ h^{-1} = h(h^{-1}(n)) = h(n+3)$. Since n is odd, then $n+3$ is even. So, $h(n+3) = (n+3)-3 = n$.
In both cases, we find that $h \circ h^{-1} = n$, which is the identity of the identity function. So, $h \circ h^{-1} = i_{\mathbb{Z}}$. So, h^{-1} is the inverse of h. □

Theorem 17.8 Discussion: We explained how $f \circ g$ implies certain things about f and g. We said if $f \circ g$ is onto (surjective), then $f(x)$ is onto, and if $f \circ g$ is one-to-one (injective), then $g(x)$ is one-to-one. But what does this say about the other functions? See **Theorem 4.**

Definitions:

Definition 1. ***Identity Function:*** *i_A is the identity function $i_A = A \to A$ and $i_A(x) = x$. It is very helpful to find inverses.*

Theorems and Proofs:

Theorem 1. ***Function Composition:*** *Two functions f and g can only be composed if the codomain of $g(x)$ is a subset of the domain of $g(x)$.*

Theorem 2. ***Inverse Function:*** *$f(x)$ has an inverse function if and only if $f(x)$ is a bijection.*

Theorem 3. ***(17.8)***

- *If $f \circ g$ is onto (surjective), then $f(x)$ is onto.*
- *If $f \circ g$ is one-to-one (injective), then $g(x)$ is one-to-one.*

Theorem 4. *If $f : A \to B$, $g : C \to A$, and $f \circ g$ is onto and f is one-to-one, then g is onto.*

- To prove this, we want to show that g is onto. We will pick $a \in A$ and show that there exists a $c \in C$ such that $g(c) = a$.
- $f \circ g = C \to B$

Proof. Let $a \in A$. Then $f(a) = b \in B$. Since f is one-to-one, a is the only element of A such that $f(a) = b$. Since $f \circ g$ is onto, we can find $c \in C$ such that $f \circ g(c) = b$. Since $f(g(c)) = b$ and a is the only element in A with $f(a) = b$, then $g(c) = a$. So there exists $c \in C$ with $g(c) = a$. Thus, g is onto. □

Note: If f is not one-to-one, then g may not be onto in this theorem. For example, $f : \{1\} \to \{1, 2\}$. $f(1) = f(2) = 1$, and $g : \{1\} \to \{1, 2\}$ $g(1) = 1$. So $f \circ g : \{1\} \to \{1\}$ $f \circ g = 1$, meaning it is onto. However, f is not one-to-one, and g is not onto.

Theorem 5. *If $f \circ g$ is one-to-one, and g is onto, then f is one-to-one.*

Independent Practice:

1. $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + y, 3x - y + 1)$ is a bijection. Find and prove its inverse function.

$g(a, b) = (\frac{a+b-1}{5}, \frac{3a-2b+2}{5}) = f^{-1}(a, b)$

Questions raised in class: We didn't have any unanswered questions. However, we started an independent practice problem that we didn't finish proving. We explained that because we know that the function is a bijection, we would only need to show $f \circ g = i_{\mathbb{R}^2}$, and simplify and show this sends $(a, b) \to (a, b)$.