MATH 267 Fall 2026 - Class Notes

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Summary: Today's class covered one-to-one (injective), onto (surjective), and bijective functions. We learned how to prove or disprove whether a function is one-to-one or onto using examples involving functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ and from $\mathbb{Z}$ to $\mathbb{Z}$

Notes:

One-to-one and Onto

- **One-to-one:** No two inputs give same output.
- **Onto:** Everything in the codomain has something mapping to it (Range and codomain are equal).

How to prove it

- **One-to-one:** Suppose $f(x_1) = f(x_2)$. Prove that $x_1 = x_2$.
- **Onto:** Show for every y in the codomain there exists x with $f(x) = y$.

How to disprove it

- **One-to-one:** Find x_1 and x_2, $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.
- **Onto:** Find a y in the codomain, show $f(x) \neq y$ for any x in the domain.

Definitions:
Definition 1. A function that is both one-to-one and onto is called a bijection.
Definition 2. If a function is onto it is called a surjection.
Definition 3. If it is one-to-one, it's called an injection.

Theorems and Proofs:

Example 1

Ex: $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x, y) = (2x + y, 3x - y + 1)$$

$$f(2,-1) = (2 \cdot 2 + (-1), 3 \cdot 2 - (-1) + 1) = (3, 8)$$

Claim: f is a bijection.

Proof: one-to-one

Suppose $(x_1, y_1) \in \mathbb{R}^2$
and $(x_2, y_2) \in \mathbb{R}^2$

$$f(x_1, y_1) = f(x_2, y_2)$$

Then

$$(2x_1 + y_1, 3x_1 - y_1 + 1) = (2x_2 + y_2, 3x_2 - y_2 + 1)$$

So

$$2x_1 + y_1 = 2x_2 + y_2$$

and

$$3x_1 - y_1 + 1 = 3x_2 - y_2 + 1$$

Adding these

$$5x_1 + 1 = 5x_2 + 1$$

Then

$$5x_1 = 5x_2$$

Thus

$$x_1 = x_2$$

Plugging this into

$$2x_1 + y_1 = 2x_2 + y_2 = 2x_1 + y_2$$

Subtracting $2x_1$ from both sides leaves $y_1 = y_2$.
So $(x_1, y_1) = (x_2, y_2)$ and f is one-to-one.

Proof: onto

Pick $(a, b) \in \mathbb{R}^2$.
We want to find $(x, y) \in \mathbb{R}^2$
with $f(x, y) = (a, b)$.

Scratch work: (Do not include this in the proof)

$$(a, b) = (2x + y, 3x - y + 1)$$

$$a = 2x + y$$

$$b = 3x - y + 1$$

$$a + b = 5x + 1$$

$$x = \frac{a + b - 1}{5}$$

$$\begin{aligned} y &= a - 2x \\ &= a - 2\left(\frac{a + b - 1}{5}\right) \\ &= \frac{5a - 2a - 2b + 2}{5} \\ &= \frac{3a - 2b + 2}{5} \end{aligned}$$

Pick

$$(x, y) = \left(\frac{a + b - 1}{5}, \frac{3a - 2b + 2}{5}\right)$$

This is in $\mathbb{R}^2$ since both entries are real numbers.
Then

$$\begin{aligned} f(x, y) = \Bigg(&2\left(\frac{a + b - 1}{5}\right) + \frac{3a - 2b + 2}{5}, \\ &3\left(\frac{a + b - 1}{5}\right) - \frac{3a - 2b + 2}{5} + 1\Bigg) \end{aligned}$$

$$= \left(\frac{2a + 2b - 2 + 3a - 2b + 2}{5}, \frac{3a + 3b - 3 - 3a + 2b - 2 + 5}{5}\right)$$

$$= \left(\frac{5a}{5}, \frac{5b}{5}\right) = (a, b)$$

Thus $f(x, y)$ is onto.
Hence f is a bijection.

Example 2

Ex: Let $h, k : \mathbb{Z} \to \mathbb{Z}$

$$h(n) = \begin{cases} n+1 & n \text{ odd} \\ n-3 & n \text{ even} \end{cases}$$

$$k(n) = \begin{cases} n+1 & n \text{ odd} \\ n-2 & n \text{ even} \end{cases}$$

Determine whether each is one-to-one or onto.

$$k(1) = 1 + 1 = 2$$

$$k(4) = 4 - 2 = 2$$

So $k(n)$ is not one-to-one.
Since $k(n)$ is even when n is even, and even when n is odd, 1 is not in the range of k.
So k is not onto.

Prove $h(n)$ is one-to-one and onto

(One-to-one)
Suppose

$$h(n_1) = h(n_2)$$

Case 1: n_1 and n_2 are both even.
Then $h(n_1) = h(n_2)$
means

$$n_1 - 3 = n_2 - 3$$

So $n_1 = n_2$.
Case 2: n_1 and n_2 are both odd then

$$n_1 + 1 = n_2 + 1$$

So $n_1 = n_2$.
Case 3: One of n_1, n_2 even, other odd.
Without loss of generality we assume n_1 is even.
Then $h(n_1) = h(n_2)$ means

$$n_1 - 3 = n_2 + 1$$

Then

$$n_1 = n_2 + 4$$

This means an even number (n_1) is 5 more than an odd number (n_2).
Since that can't happen, this case cannot happen.
Thus $n_1 = n_2$ in the only cases that are allowed, so $h(n)$ is one-to-one.

2) Proving onto

Pick $m \in \mathbb{Z}$.
Show there exists $n \in \mathbb{Z}$ with $h(n) = m$.
Case 1: m is odd.
Pick $n = m + 3$.
Then n is even, so

$$h(n) = h(m + 3) = m + 3 - 3 = m$$

Case 2: m is even.
Pick $n = m - 1$.
Then n odd so

$$h(n) = h(m - 1) = (m - 1) + 1 = m$$

Thus $h(n)$ is onto hence a bijection.

Questions raised in class: No unresolved questions were recorded in my notes.