MATH 267 Fall 2026 - Class Notes

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Summary:

Notes: We began class by finishing the problem from Monday's class that we were unable to finish.

A = $\{\frac{1}{m} + \frac{1}{n} : m,n \in \mathbb{Z}_{>0}\}$

Find max(A), up(A), inf(A), and min(A).

Max(A) = sup(A) = 2

$\frac{1}{m} \leq 1/1 = 1$

$\frac{1}{n} \leq 1/1 = 1$

so $\frac{1}{m} + \frac{1}{n} \leq 1$

So, 2 is an upper bound for A.

Since m=1, n=1 gives $\frac{1}{1} + \frac{1}{1} = 2 \in A$

2 is the maximum of A, so it is also the supremum.

Note that $\frac{1}{m} > 0$ for any $m \in \mathbb{Z}_{>0}\}$ Similarly, $\frac{1}{n} > 0$.

So, $\frac{1}{m} + \frac{1}{n} > 0 + 0 = 0$

0 is a lower bound for A, however we still need to show that 0 is the infemum of A.

Suppose y > 0 is also a lower bound for A.

Scratch work:

We need m and n such that $\frac{1}{m} + \frac{1}{n} < y$

Goal: find m such that $\frac{1}{m} < y/2$ set n=m

Tool: archimedean property

for any positive real numbers a,b we can find $n \in \mathbb{Z}_{>0}\}$ such that $a < bn$

We can always find an n to make b bigger than a.

$\frac{1}{m} < \frac{y}{2}$ multiply both sides by m

$1 < m\frac{y}{2}$ or $2 < my$

By the archimedean property, there exists an integer m such that $1 < m\frac{y}{2}$ setting $\frac{1}{m} + \frac{1}{n} < \frac{y}{2} + \frac{y}{2} = y$

So, y is not the lower bound for A.

But 0 was, so 0 is the **infemum** of A.

Since 2 is the smallest possible upper bound, anything smaller would be in the set.

What if we swapped + for -? B = $\{\frac{1}{m} - \frac{1}{n} : m,n \in \mathbb{Z}_{>0}\}$

Conjecture:

inf(B) = -1

max(B) = DNE

min(B) = DNE

sup(B) = 1

Warm-up Examples of Functions:

A = {1,2,3,4} B = {p,q,r}

R_1 = {(1,p),(2,q),(3,p),(4,r)} Yes, this is a function.

R_2 = {(1,p),(2,q),(4,r)} No, this is not a function. Reason: 3 is not matched with any element in B.

R_3 = {(1,p),(2,q),(3,r),(4,q),(2,r)} No, this is not a function. Reason: 2 has an ambiguous output (has been matched with 2 elements in B).

R_4 = {(1,p),(2,q),(3,q),(4,q)} Yes, this is a function.

Applications of range, domain, and codomain in the warmup examples:

R_1 = {(1,p),(2,q),(3,p),(4,r)}

Domain(R_1) = A

Codomain(R_1) = B

Range(R_1) = B

R_4 = {(1,p),(2,q),(3,q),(4,q)}

Domain(R_4) = A

Codomain(R_4) = B

Range(R_4) = {p,q}

Example 1: F: $[0, \infty) \to \mathbb{R}$

f(x) = $\frac{x}{x+1}$

Find the domain, codomain, and range of f:

Domain: $[0, \infty)$

Codomain: $\mathbb{R}$

However, the range requires more scratch work:

y = $\frac{x}{x+1}$ = $\frac{x+1-1}{x+1}$ = $\frac{x+1}{x+1}$ - $\frac{1}{x+1}$ = 1 - $\frac{1}{x+1}$

y - 1 = -$\frac{1}{x+1}$

x + 1 = -$\frac{1}{y-1}$

x = -$\frac{1}{y-1}$ - 1

x = $\frac{1}{1-y}$ - 1

x = $\frac{1-(1-y)}{1-y}$

$\frac{y}{1-y}$ = x

Proof that ran(f) = $[0, \infty)$ using double inclusion

($\subseteq$): Let y $\in$ Ran(f)

So y = $\frac{1}{x+1}$ for some x ≥ 0

Both x ≥ 0 and x + 1 > 0

So $\frac{x}{x+1} > 0$

Also y = $\frac{x}{x+1} = \frac{x+1-1}{x+1}$

Which is equal to 1 - $\frac{1}{x+1} < 0$ (since x + 1 > 0)

$0 \leq$ y < 1

y $\in [0, 1)$ and ran(f) $\subseteq [0, 1)$

($\supseteq$): Let y $\in [0, 1)$

Let x = $\frac{y}{1-y}$

Since $0 \leq$ y < 1

1 - y > 0

So x > 0 is in the domain of F

evaluate f(x) for this x

$$\mathrm{f(x)} = \mathrm{f}\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{\frac{y}{1-y}+1} = \frac{\frac{y}{1-y}}{\frac{y}{1-y}+\frac{1-y}{1-y}} = \frac{\frac{y}{1-y}}{\frac{1}{1-y}}$$

multiply the numerator and denominator by (1-y) gives

$$\frac{\frac{y}{1-y}(1-y)}{\frac{y}{1-y}(1-y)}$$

So y $\in$ Ran(f) $\quad [0, \infty) \subseteq Ran(f)$

So $[0, \infty)$ = Ran(f)

Example 2: Define F: $\mathbb{Q} \to \mathbb{Q}$ and G: $\mathbb{Q} \to \mathbb{Q}$

By $\mathrm{F}\left(\frac{a}{b}\right) = \frac{a^2}{b^2+a^2}$ $\mathrm{G}\left(\frac{a}{b}\right) = \frac{a+1}{b+1}$

Claim one of these is a function and one isn't

Note: $\mathrm{G}\left(\frac{1}{-1}\right) = \frac{1+1}{-1+1} = \frac{2}{0}$ the output is not defined, and in order to be a valid function the output must be defined

Definitions:

Definition 1. *A **function** f: A $\to$ B is a rule that assigns to every a $\in$ A exactly ONE b $\in$ B.*

Ways a relation can fail to be a function:

1. some a $\in$ A is not given a match
2. some a $\in$ A is paired with something outside B
3. the output for some a $\in$ A is ambiguous

Definition 2. *The **domain** of a function is the set of objects the function maps from (everything in the domain has an output)*

Definition 3. *The **codomain** of a function is the set of things the function maps to (a function may not map to everything in the codomain)*

Definition 4. *The **range** is the set of things in the codomain that are the output of some element of the domain*

ran(f) = {b ∈ codomain(f): f(a) = b for some b ∈ domain(f)}

Theorems and Proofs: No theorems covered in this lecture.

Questions raised in class: