MATH 267 Fall 2026 - Class Notes

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Summary: Today's class covered upper bounds, maximums, supremums, and the Completeness Axiom of real numbers. We also learned about infimums, minimums, and the Archimedean Property through examples and proofs.

Notes:

Upper Bound

$U \subseteq \mathbb{R}$

An upper bound for U is larger than or equal to every element of U.

- Maximum of U (y): y is element of U and no element of U is larger than y.
- Supremum of U (y): y is larger than or equal to every element of U, and no smaller number has these property.

There can be a lot of upper bound.

How would you disprove for y

1. Find $s \in U$ such that $s > y$.
2. Either $y \notin U$ or find $s \in U$, $s > y$.
3. Either find $s \in U$, $s > y$ or find $w < y$, no element of U is larger than w.

Warm up

A. $(-2, 5]$

B. $\{u^2 : u \in \mathbb{Z}\}$

C. $\left\{\frac{1}{x} : x \in \mathbb{R},\ x^2 < x + 2\right\}$

D. $\{-u^4 : u \in \mathbb{Z}\}$

E. $(-\infty, 1) \cup \{2\}$

	An upper bound	Max	Sup
A	6	5	5
B	none	none	none
C	100	2	2
D	none	none	none
E	2	2	2

B, D goes to infinity so they don't have an upper bound, max or sup.

Definitions:

- An upper bound for U is larger than or equal to every element of U.
- Maximum of U (y): y is element of U and no element of U is larger than y.
- Supremum of U (y): y is larger than or equal to every element of U, and no smaller number has these property.

Theorems and Proofs:

Completeness Axiom

If $S \subseteq \mathbb{R}$ has an upper bound then S has a supremum.

(Fact about Real numbers)

Note: this is not true for $\mathbb{Q}$.

$$\{u \in \mathbb{Q} : u^2 < 2\}$$

has an upper bound in $\mathbb{Q}$ (3/2).

No supremum in $\mathbb{Q}$ (there is one in $\mathbb{R}$ namely $\sqrt{2}$).

Example

Ex:

$$T = \{x^2 - 3 : x \in \mathbb{R},\ x > 1\}$$

Claim: the infimum of T is -2 and it has no minimum.

Proof:

Since $x > 1$, then $x^2 > x > 1$ so

$$x^2 - 3 > 1 - 3 = -2$$

For all $x > 1$ so -2 is a lower bound for T.

To show that -2 is the infimum, we need to show that nothing larger than it is a lower bound for T.

Suppose for contradiction that $y > -2$ were a lower bound for T.

Write

$$x = 1 + t$$

Set

$$t = \min\left\{1, \frac{y+2}{8}\right\}$$

Write $x = 1 + t$ with $t > 0$.

$$\begin{aligned} x^2 - 3 &= (1+t)^2 - 3 \\ &= 1 + 2t + t^2 - 3 \\ &= -2 + 2t + t^2 \end{aligned}$$

If $t < 1$, $t^2 < t$, then

$$-2 + 2t + t^2 < -2 + 2t + t$$

$$= -2 + 3t$$

Want

$$-2 + 3t < y$$

so

$$y + 2 > 3t$$

$$\frac{y+2}{3} > t$$

Picking

$$t = \min\left\{1, \frac{y+2}{8}\right\}$$

guarantees both that $t \leq 1$ and $t \leq \frac{y+2}{8}$.

Since both 1 and $\frac{y+2}{8} > 0$,

$$1 + t > 1$$

is a potential value of x in T.

But

$$\begin{aligned} x^2 - 3 &= (1+t)^2 - 3 \\ &= 1 + 2t + t^2 - 3 \\ &\leq -2 + 3t \quad (\text{Since } t \leq 1) \\ &\leq -2 + 3\left(\frac{y+2}{8}\right) \quad \left(\text{Since } t \leq \frac{y+2}{8}\right) \\ &= -2 + \frac{3y}{8} + \frac{6}{8} \\ &= -2 + \frac{3(y+2)}{8} \\ &< -2 + (y+2) \\ &= y \end{aligned}$$

We found a smaller value in T than y.

We found an x value $x^2 - 3 < y$.

So y cannot be a lower bound for T, since -2 is a lower bound but no $y > -2$ is a lower bound, -2 is an infimum for T.

-2 is not in T so -2 is not a minimum of T.

So T has no minimum but -2 is its infimum.

The Archimedean Property

If a and b are real numbers then we can find an integer $n \in \mathbb{Z}$ such that

$$a < b \cdot n$$

(equivalently $\frac{a}{b} < n$)

For any real number we can always find a larger integer.

Example

Ex:

$$S = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}^+ \right\}$$

What are the Inf(S), Min(S), Max(S), Sup(S)?

Note

$$\frac{1}{m} > 0 \quad \text{and} \quad \frac{1}{n} > 0$$

so

$$\frac{1}{m} + \frac{1}{n} > 0$$

So 0 is a lower bound.

Questions raised in class: No unresolved questions were recorded in my notes.