MATH 267 Fall 2026 - Class Notes

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Summary:
We began class with a warm up questions from chapter 13. We had to identify Upper bounds, Maximum, Supremum or least upper bound, Lower bounds, minimum, and infimum (or greatest lower bound) of the sets. Then, we started chapter 13 with sub-topic Upper Bounds. We did few examples with a prove. The lecture ended while the Archimedean property was being presented.

Notes:
Today's class was about Chapter 13; Order in the Reals. We discussed about Upper Bounds. $U \subseteq \mathbb{R}$.

1. An Upper bound for U implies No element of U is larger than Y. Y is larger than every element of U.
2. Maximum of U implies Y is element of U and no element of U is larger than Y.
3. Supremum of U implies No element of U is larger than Y and no smaller number has this property

How would you Disapprove for Y?

- Find $s \in U$ such that $s > Y$
- Either $y \notin U$ or find $s \subseteq U$ $s > Y$
- Either find $s \in U$, $s > Y$ or find $w < Y$. This implies that no element in U is larger than W.

Warm-up Review

	Upper Bound	Max	Supremum	Lower Bounds	Min	Infimum
A= (-2, 5] ⇒	6	5	5	-10	None	-2
B= $\{n^2 : n \in \mathbb{Z}\}$ ⇒	None	None	None	-1	0	0
C= $\{x \in \mathbb{R} : x^2 \leq x + 2\}$ ⇒	100	2	2	-1	-1	-1
D= $(-1)^n n : n \in \mathbb{Z}$ ⇒	None	None	None			
E= $[0, 1] U \{2\}$ ⇒	2	2	2			

Completeness Axiom
If $s \subseteq \mathbb{R}$ has an upper bound, then s has a Supremum. (Facts about Real Numbers).
Note: This is not true for $\mathbb{Q}$ $\{x \in \mathbb{Q} : x^2 < 2\}$. Has an Upper Bound in $\mathbb{Q}(3/2)$. No Supremum in $\mathbb{Q}$ (there is one in $\mathbb{R}$ namely $\sqrt{2}$).
Example: $T = \{x^2 - 3 : x \in \mathbb{R} x > 1\}$.

Claim: The infimum of T is -2 and it has no minimum.
Proof: Since $x > 1$, $x^2 > x > 1$ so $x^2 - 3 > 1 - 3 = -2$ for all $x > 1$. So -2 is a lower bound for T.
To show that -2 is the infimum we need to show that nothing larger is a lower bound for T. Suppose for contradiction that $y > -2$ a lower bound of T. Write $x = 1 + t$ with $t > 0$.
$X^2 - 3 \Rightarrow (1+t)^2 - 3 = 1 + 2t + t^2 - 3 = -2 + 2t + t^2$
If $t \leq 1$, $t^2 \leq t$ then $-2 + 2t + t^2 \leq -2 + 2t + t = -2 + 3t$.
We want $-2 + 3t < y$ so $y + 2 > 3t \Rightarrow \frac{y+2}{3} > t$. Picking $t = Min\{1, \frac{y+2}{6}\}$ guarantees both that $t \leq 1$ and $t \leq \frac{y+2}{6}$. Since both 1 and $\frac{y+2}{6} > 0$, $x = 1 + t > 1$ is a potential value of x in T. but $x^2 - 3 = (1+t)^2 - 3 = 1 + 2t + t^2 - 3 \leq -2 + 2t + t = -2 + 3t \leq -2 + 3(\frac{y+2}{6}) = -2 + \frac{y+2}{6} < -2 + (y+2) = y$. We found a smaller value in T than Y. We found an x value $x^2 - 3 < y$ so Y cannot be a lower bound of T Since -2 is a lower bound but no $y > -2$ is a lower bound, Y is an infimum for T. $-2 \notin T$ so -2 is not a minimum of T. So T has no minimum but -2 is its infimum.
The Archimedean Property
If a and b are real numbers, then we can find an integer $n \in \mathbb{Z}$ such that $a < bn$ (equivalently $\frac{a}{b} < n$). For any real number we can always find a large integer.
Example:
$S = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}^+\}$
What are Infimum(S), Minimum(S), Maximum(S), Supremum(S)?
Note: $\frac{1}{m} > 0$ and $\frac{1}{n} > 0$ so $\frac{1}{m} + \frac{1}{n} > 0$ so 0 is a lower bound.

Definitions:

1. An Upper bound for U implies No element of U is larger than Y. Y is larger than every element of U.

2. Maximum of U implies Y is element of U and no element of U is larger than Y.

3. Supremum of U implies No element of U is larger than Y and no smaller number has this property

4. Completeness Axiom. If $s \subseteq \mathbb{R}$ has an upper bound, then s has a Supremum. (Facts about Real Numbers).