MATH 267 Fall 2026 - Class Notes

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Scribe: Reggie Carbajal

Notes

Definition of a Partition A partion of a set X is a collection of subsets that satisfies three conditions:

1. Every part must be noempty
2. The union of all the parts must equal X
3. Any two parts must be either disjoint or equal.

Two sets are disjoint if they have no elements in common.Two sets are equal if they contain exactly the same elements.

Warm-up Problem 1

Let

$$X = \{1, 2, 3, 4, 5, 6\}.$$

Determine if each collection is a partittion of X.

A.

$$A - \{1, 2\}, \{3, 4\}, \{5, 6\}\}.$$

Yes, A is a partition because all the parts are nonempty,they cover X, and they do not overlap.

B.

$$B = \{\{1, 2, 3\}, \{3, 4\}, \{5, 5\}\}.$$

No, B is not a partition because 3 appears in two different parts that are not equal.

C.

$$C = \{\{1, 3\}, \{4, 5, 6\}\}.$$

No,C is not a partition because 2 is missing.The union does not equal X.

D.

$$D = \{\{1, 2, 3\}, \emptyset, \{4, 5, 6\}\}.$$

No, D is not a partition because it contains the empty set.

E.

$$E = \{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{4, 1\}\}.$$

Yes, E is a partition because $\{1, 4\}$ and $\{4, 1\}$ are equal sets. Repeating the same set does not create a different part.

Example 1: Infinite Partition

For every odd positive integer m, define

$$A_m = \{m, 2m, 4m, 8m, \ldots\}.$$

We can also write

$$A_m = \{2^i m : i = 0, 1, 2, \ldots\}.$$

Let

$$S = \{A_m : m \text{ is an odd positive integer}\}.$$

Some examples are

$$A_1 = \{1, 2, 4, 8, 16, 32, \ldots\},$$
$$A_3 = \{3, 6, 12, 24, 48, \ldots\},$$
$$A_5 = \{5, 10, 20, 40, 80, \ldots\}.$$

We want to show that S is a partition of $\mathbb{Z}^+$. **1. Every part is nonempty.** Each A_m contains m because $2^0 m = m$. Therefore, every part is nonempty.

2. The union equals $\mathbb{Z}^+$.

Every positive integer n can be written as

$$n = 2^k m,$$

where m is odd and $k \geq 0$.

This means every positive integer belongs to one of the sets A_m.

Also, every element of A_m is a positive integer. Therefore,

$$\bigcup_{\substack{m \geq 1 \\ m \text{ odd}}} A_m = \mathbb{Z}^+.$$

3. The parts are disjoint or equal.

Suppose A_m and $A_{m'}$ share an element n.

Then

$$n = 2^k m = 2^j m'$$

for some integers $k, j \geq 0$.

Assume $k \geq j$. Dividing by 2^j gives

$$2^{k-j} m = m'.$$

Since both m and m' are odd, we must have

$$k - j = 0.$$

Therefore,

$$m = m'$$

and

$$A_m = A_{m'}.$$

So the parts are either disjoint or equal. Since all three conditions hold, S is a partition of $\mathbb{Z}^+$.

Partitions and Equivalence Relations

Every equivalence relation gives a partition ,and every partition gives an equivalence relation. The equivalence classes of an equivalence relation form a partition. If we have a partition, we can define a relation by saying two elements are related if they belong to the same part. Using our partition of positive integer, two numbers are related when they have the same odd part. For example,

$$5 \sim 40, \qquad 9 \sim 36, \qquad 12 \sim 48.$$

However,

$$10 \not\sim 100,$$

because the odd part of 10 is 5, while the odd part of 100 is 25.

Warm-up Problem 2: Number of Divisors

Let

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Define

$$x \sim y$$

if x and y have the same number of positive divisors.

The number of divisors for each number is:

n	Number of divisors
1	1
2	2
3	2
4	3
5	2
6	4
7	2
8	4
9	3
10	4

We group numbers that have the same number of divisors.

The partition is

$$\{\{1\}, \{2, 3, 5, 7\}, \{4, 9\}, \{6, 8, 10\}\}.$$

The set $\{1\}$ contains the number with one divisor.

The set $\{2, 3, 5, 7\}$ contains numbers with two divisors.

The set $\{4, 9\}$ contains numbers with three divisors.

The set $\{6, 8, 10\}$ contains numbers with four divisors.

Writing out the equivalence relation as ordered pairs would take more work, but writing the partition is easier.

Example 3: Equivalence Classes

Define a relation on $\mathbb{Z}$ by

$$x \sim y \iff 4 \mid (x + 3y).$$

We want to list the equivalence classes.

Start with $x = 1$ and try different values of y.

$$\begin{aligned} y = 1 &: 1 + 3(1) = 4, \\ y = 2 &: 1 + 3(2) = 7, \\ y = 3 &: 1 + 3(3) = 10, \\ y = 4 &: 1 + 3(4) = 13, \\ y = 5 &: 1 + 3(5) = 16, \\ y = 9 &: 1 + 3(9) = 28, \\ y = 13 &: 1 + 3(13) = 40. \end{aligned}$$

The values 4, 16, 28, and 40 are divisible by 4.

So 1 is related to

$$1, 5, 9, 13, 17, 21, \ldots$$

These values follow the pattern

$$y = 1 + 4k$$

where $k \in \mathbb{Z}$.

Substituting gives

$$1 + 3(1 + 4k) = 4 + 12k = 4(1 + 3k).$$

This is divisible by 4.

The equivalence classes are

$$E_1 = \{4k + 1 : k \in \mathbb{Z}\},$$

$$E_2 = \{4k + 2 : k \in \mathbb{Z}\},$$

$$E_3 = \{4k + 3 : k \in \mathbb{Z}\},$$

$$E_4 = \{4k + 4 : k \in \mathbb{Z}\}.$$

These four classes form a partition of $\mathbb{Z}$.

Final Classroom Problem

Near the end of class, we started another problem but did not finish it.

On

$$\mathcal{P}(\{1, 2, 3\}),$$

define

$$A \sim B \iff A \cap \{1, 2\} = B \cap \{1, 2\}.$$

We were asked to list the equivalence classes.

Some of the examples were

$$\{\emptyset, \{3\}\},$$

$$\{1\} \sim \{1, 3\},$$

and

$$\{2\} \sim \{2, 3\}.$$

The idea is to group that have the same intersection with $\{1, 2\}$. We did not finish the problem during class.

Definitions

Partition: A collection of nonempty subsets whose union equals the original set, where any two parts disjoint or equal. **Disjoint sets:** Two sets that contain exactly the same elements. **Disjoint sets:** Two sets that have no elements in common.

Equal sets: Two sets that contain exactly the same elements.

Equivalence relation: A relation that is reflexive, symmetric, and transitive.

Equivalence class: A set containing all elements related to a particular element.

Theorems and Proofs

Theorem 11.4: Every equivalence relation on a set determines a partition of that set, and every partition determines an equivalence relation.