

# MATH 267 Fall 2026 - Class Notes

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## Summary:

We started with problem 1 of the warm-up. Then we discussed partitions and the definitions of disjoint and equal in the context of partitions. Near the end of class, we went over problem 2 of the warm-up. Lastly, we had two problems to work on our own, and were able to work through the first problem as a class, but got cut short during the second.

The next written homework (Written Assignment 4) will be assigned soon.

Midterm exams should be graded and returned Monday.

## Notes:

Warm-up example:

Let  $X = \{1,2,3,4,5,6\}$  Determine if each of the following sets are partitions of  $X$ .

$A = \{\{1,2\}, \{3,4\}, \{5,6\}\}$  Yes -  $A$  is a partition.

$B = \{\{1,2,3\}, \{3,4\}, \{5,6\}\}$  No -  $B$  is not a partition. Reason: 3 appears in two separate sets

$C = \{\{1,3\}, \{4,5,6\}\}$  No -  $C$  is not a partition. Reason: 3 is not an element in any set.

$D = \{\{1,2,3\}, \emptyset, \{4,5,6\}\}$  No -  $D$  is not a partition. Reason: the empty set cannot be included, as the criteria to be a partition states each element must be nonempty.

$E = \{\{1,4\}, \{2,5\}, \{3,6\}, \{4,1\}\}$  Yes -  $D$  is a partition. Reason:  $\{1,4\}$  and  $\{4,1\}$  are not disjoint, they are equal because elements can be repeated if the whole set repeats.

**Example 1:** Let  $A_m = \{m, 2m, 4m, 6m, \dots\} = \{2im \mid i \in \mathbb{Z}^+\}$

For each odd positive integer  $m$ , then the collection of  $S = \{A_m : m \text{ odd}, m \in \mathbb{Z}^+\}$  forms a partition of  $\mathbb{Z}^+$

$$A_1 = \{1, 2, 4, 8, 16, 32, \dots\}$$

$$A_3 = \{3, 6, 12, 24, 48, \dots\}$$

$$A_5 = \{5, 10, 20, 40, 80, \dots\}$$

Both the set and the partition are infinite.

Proof (that this is a partition):

1. Each  $A_m \in S$  is nonempty since  $m \in A_m$

2.  $\bigcup_{\text{odd } m} A_m = \mathbb{Z}^+$  for every  $m$   $\bigcup_{\text{odd } m} A_m \subseteq \mathbb{Z}^+$

Every positive integer  $n$  can be written as  $n = 2^k m$  for some odd integer  $m$  and  $k \geq 0$ .

For any  $n \in \mathbb{Z}^+$ , write  $n = 2^k m$  when  $m$  is odd, then  $n \in A_m$ , so  $\mathbb{Z}^+ \subseteq \bigcup_{\text{odd } m} A_m$

Thus,  $\bigcap_{\text{odd } m} A_m = \mathbb{Z}^+$ , (showing double containment)

3. Show  $A_m$  are disjoint or equal

Proof by contradiction.

Suppose there exists  $A_m$  and  $A_{m'}$  such that they are not disjoint and not equal. (not disjoint meaning their intersection is not empty)

Then  $\exists n \in A_m, n \in A_{m'}$

Thus,  $n = 2^k m = 2^j m'$  for some integers  $k$  and  $j$ . We can assume  $k \geq j$ , and if not we will swap  $m$  and  $m'$  to make  $k$  the bigger of the two.

Since  $2^k m = 2^j m'$  dividing by  $2^j$  will result in  $2^{k-j} m = m'$ . (since  $k \geq j$ ,  $2^{k-j}$  will be a positive integer)

Since  $m'$  is odd,  $k - j = 0$ . (otherwise  $m'$  would be divisible by 2)

$$2^{k-j} = 2^0 = 1$$

So  $1m = m'$ , then  $A_m = A_{m'}$

So then  $A_m$  forms a partition of  $\mathbb{Z}^+$

Every equivalence relation gives us a partition (the equivalence classes form a partition and every partition gives us an equivalence relation)

If  $S$  is a partition of  $X$ , define an equivalence relation  $\sim_S$  or  $R_S$  by  $A \sim_S B$  if and only if  $a$  and  $b$  are in the same part of the partition.

Using our partition of  $\mathbb{Z}^+$  into  $A_m$  we get an equivalence relation  $\sim$  where  $n \sim n'$  if and only if  $n$  and  $n'$  have the same odd part.

Examples:  $5 \sim 40$ ,  $9 \sim 36$ ,  $12 \sim 48$

$10 \sim 100$  does not work, because the odd integer of 10 is 5, and the odd integer of 100 is 25.

### Example 2 (warm-up problem 2):

Let  $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$x \sim y$  if  $x$  and  $y$  have the same number of divisors.

The partition of  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  is  $\{\{1\}, \{2, 3, 5, 7\}, \{4, 9\}, \{6, 8, 10\}\}$

The set of  $\{1\}$  has elements with 1 divisor The set of  $\{2, 3, 5, 7\}$  has elements with 2 divisors The set of  $\{4, 9\}$  has elements with 3 divisors The set of  $\{6, 8, 10\}$  has elements with 4 divisors

Partition = equivalence relation

$\{(a, a) \text{ for every } a\} \cup \{(2, 3), (2, 5), (2, 7), (3, 5), (3, 7), (5, 7), (3, 2), (5, 2), (7, 2), (5, 3), (7, 3), (7, 5), \dots\}$

Writing out the equivalence relation as ordered pairs is tedious, but writing out the partition is relatively easy.

Table 1: n and the number of integer divisors

| n  | number of divisors |
|----|--------------------|
| 1  | 1                  |
| 2  | 2                  |
| 3  | 2                  |
| 4  | 3                  |
| 5  | 2                  |
| 6  | 4                  |
| 7  | 2                  |
| 8  | 4                  |
| 9  | 3                  |
| 10 | 4                  |

4. **Example 3:** Define an equivalence relation on  $\mathbb{Z}$  by  $x \sim y$  if and only if  $4 \mid (x + 3y)$

List the equivalence classes of  $\sim$  with a few elements in each

Work through  $x=1$  for various values of  $y$ :

$$y = 1: 1 + 3(1) = 1 + 3 = 4 \text{ (divisible by 4)}$$

$$y = 2: 1 + 3(2) = 1 + 6 = 7$$

$$y = 3: 1 + 3(3) = 1 + 9 = 10$$

$$y = 4: 1 + 3(4) = 1 + 12 = 13$$

$$y = 5: 1 + 3(5) = 1 + 15 = 16 \text{ (divisible by 4)}$$

$$y = 6: 1 + 3(6) = 1 + 18 = 19$$

$$y = 7: 1 + 3(7) = 1 + 21 = 22$$

$$y = 8: 1 + 3(8) = 1 + 24 = 25$$

$$y = 9: 1 + 3(9) = 1 + 27 = 28 \text{ (divisible by 4)}$$

$$y = 10: 1 + 3(10) = 1 + 30 = 31$$

...

$$y = 13: 1 + 3(13) = 1 + 39 = 40 \text{ (divisible by 4)}$$

$x = 1$  is related to  $y = 1, 5, 9, 13, \dots$  which are integers of  $y$  for some integer  $k$  such that  $1 + 4k$ .

$$x = 1, y = 1 + 4k$$

$$\text{Sub in: } 1 + 3(1 + 4k) = 1 + 12k + 3 = 12k + 4 = 4(3k + 1)$$

equivalence class of  $x \sim y \iff 4 \mid (x + 3y)$

$$E_1 = \{1, 5, 9, 13, 17, 21, \dots\}$$

$$E_2 = \{2,6,10,14,18,22,\dots\} \quad (4k+1)$$

We can see the pattern form and find that the equivalence classes are as follows:

$$E_1 = \{4k+1 : k \in \mathbb{Z}\}$$

$$E_2 = \{4k+2 : k \in \mathbb{Z}\}$$

$$E_3 = \{4k+3 : k \in \mathbb{Z}\}$$

$$E_4 = \{4k+4 : k \in \mathbb{Z}\}$$

$E_1, E_2, E_3, E_4$  are all partitions of  $\mathbb{Z}$ .

### Definitions:

**Definition 1.** A partition is a collection of subsets  $S \subseteq P(x)$  if

1. each element of  $S$  is nonempty

2.  $\bigcup_{s \in S} S = X$

Every element of  $S$  is in one of the parts with no common elements (unless an entire set is repeated)

3. each element of  $S$  is either equal to or disjoint from all others

**Theorems and Proofs:** Use theorem environments for important results:

**Theorem 1.**

*Proof.*

□

### Questions raised in class:

Only worked on for a few minutes at the end of class - we were not able to work very much on this example.

On  $P(\{1,2,3\})$  define  $A \sim B$  if and only if  $A \cap \{1,2\} = B \cap \{1,2\}$

List the equivalence classes.

$$\{A \in P(\{1,2,3\}) : A \cap \{1,2\} = \emptyset\} = \{\emptyset, \{3\}\}$$

$$\{1\} \sim \{1,3\}$$

$$\{2\} \sim \{2,3\}$$

$$\{3\} \sim \{1,2,3\}$$