MATH 267 Fall 2026 - Class Notes

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Summary

In this class we learned what a relation is and how to check whether a relation is reflexive, symmetric, or transitive. We used three relations on the set $A = \{1, 2, 3, 4\}$ to practice checking these properties. We also introduced equivalence relations and how they divide a set into equivalence classes.

Notes

Relations. A relation R on a set A is a subset of $A \times A$. This means that a relation is a collection of ordered pairs whose entries come from A. We can write $a \sim b$ to mean $(a, b) \in R$.

Three properties of relations.

- **Reflexive:** Every element must be related to itself. For $A = \{1, 2, 3, 4\}$, we need $(1, 1), (2, 2), (3, 3), (4, 4)$ in the relation.
- **Symmetric:** Whenever (a, b) is in the relation, the reversed pair (b, a) must also be in the relation. For example, if $(1, 3) \in R$, we must also have $(3, 1) \in R$.
- **Transitive:** If (a, b) and (b, c) are in the relation, then (a, c) must be in the relation too. For example, $(1, 2)$ and $(2, 3)$ require $(1, 3)$.

Examples from Class

Example 1: Rational Quotient

Let $A = \mathbb{R} \setminus \{0\}$.

Define a relation R on A by

$$(a, b) \in R \iff \frac{a}{b} \in \mathbb{Q}.$$

We want to show that this is an equivalence relation.

Reflexive: For every $a \in A$,

$$\frac{a}{a} = 1 \in \mathbb{Q}.$$

Therefore, R is reflexive.

Symmetric: Suppose $(a, b) \in R$. Then

$$\frac{a}{b} \in \mathbb{Q}.$$

Since a and b are nonzero,

$$\frac{b}{a} = \left(\frac{a}{b}\right)^{-1} \in \mathbb{Q}.$$

Therefore, R is symmetric.

Transitive: Suppose $(a, b) \in R$ and $(b, c) \in R$. Then

$$\frac{a}{b} \in \mathbb{Q}, \qquad \frac{b}{c} \in \mathbb{Q}.$$

Multiplying gives

$$\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} \in \mathbb{Q}.$$

Therefore, R is transitive.

Since all three properties hold, R is an equivalence relation.

Example 2: Divisibility

Let $A = \mathbb{Z}^+$.

Define a relation R by

$$(a, b) \in R \iff a \mid b.$$

Reflexive: Yes, because every positive integer divides itself. For example,

$$a = a \cdot 1.$$

Symmetric: No, because

$$2 \mid 6$$

but

$$6 \nmid 2.$$

Transitive: Yes, because if $a \mid b$ and $b \mid c$, then $a \mid c$.

Therefore, this is not an equivalence relation because it is not symmetric.

We used the set

$$A = \{1, 2, 3, 4\}.$$

The three relations written on the board were

$$\begin{aligned}
R_1 &= \{(1,1), (2,2), (3,3), (4,4), (1,3), (3,1)\}, \\
R_2 &= \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), \\
&\qquad (2,3), (3,2)\}, \\
R_3 &= \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,3), (1,3)\}.
\end{aligned}$$

- R_1: It is reflexive because all four pairs (a, a) are included. It is symmetric because $(1, 3)$ and $(3, 1)$ are both included. It is also transitive: for example, $(1, 3)$ and $(3, 1)$ require $(1, 1)$, which is included. The other required pairs are included as well.
- R_2: It is reflexive and symmetric. It is not transitive because $(1, 2) \in R_2$ and $(2, 3) \in R_2$, but $(1, 3) \notin R_2$.
- R_3: It is reflexive. It is not symmetric because $(1, 2) \in R_3$ but $(2, 1) \notin R_3$. It is transitive because $(1, 2)$ and $(2, 3)$ give $(1, 3)$, and the required pairs involving self-pairs are present.

Summary of the Class Examples

	Reflexive	Symmetric	Transitive
R_1	Yes	Yes	Yes
R_2	Yes	Yes	No
R_3	Yes	No	Yes

Equivalence Relations and Classes

A relation is an equivalence relation when it has all three properties: reflexive, symmetric, and transitive. Of the three examples, only R_1 is an equivalence relation.

An equivalence relation separates the set into equivalence classes, where elements in the same class are related. For R_1, the equivalence classes are

$$\{1, 3\}, \qquad \{2\}, \qquad \{4\}.$$

Here 1 and 3 belong together because $(1, 3)$ and $(3, 1)$ are in R_1. The elements 2 and 4 are each in their own class. Each element of A belongs to one of these classes.

Additional Example (For Practice)

Let $B = \{1, 2, 3\}$ and

$$S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}.$$

This relation is reflexive because all three self-pairs are included. It is symmetric because $(1, 2)$ and $(2, 1)$ both appear. It is transitive: the only connections between different elements are 1 and 2, and combining them gives self-pairs already in S.

Thus S is an equivalence relation, with classes $\{1, 2\}$ and $\{3\}$. This extra example shows how we can check the same three properties on a smaller set.

Definitions

Definition 1 (Relation). A relation R on a set A is a subset of $A \times A$.

Definition 2 (Reflexive). A relation R on A is reflexive if $(a, a) \in R$ for every $a \in A$.

Definition 3 (Symmetric). A relation R on A is symmetric if $(a, b) \in R$ implies $(b, a) \in R$.

Definition 4 (Transitive). A relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$.

Definition 5 (Equivalence Relation). A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

Equivalence classes: An equivalence relation partitions A into equivalence classes: subsets that group related elements together.