

MATH 267 Fall 2026 - Class Notes

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Summary: This lecture introduced relations (Chapter 11) on sets and identified the properties required for an equivalence relation: reflexivity, symmetry, and transitivity. We worked through several examples evaluating whether specific relations such as divisibility and absolute difference satisfy the properties. We also discussed how equivalence relations partition a set into distinct equivalence classes consisting of elements related to one another.

Notes:

From Warm-up: $A = \{1, 2, 3, 4\}$

$$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 2)\}$$

$$R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3)\}$$

Reflexive:

All are reflexive.

Symmetric:

R_1 and R_2 are symmetric.

R_3 is not because $(1, 2), (2, 3), (1, 3)$ are present, but $(2, 1), (3, 2), (3, 1)$ are not.

Transitive:

R_1 and R_3 are transitive.

R_2 is not because $(1, 2) \sim (2, 1) \rightarrow (1, 1)$

but $(1, 2) \sim (2, 3) \rightarrow (1, 3)$, which is not present in the relation.

R_1 is an equivalence relation.

Equivalence Classes: $\{1, 3\}, \{2\}, \{4\}$

Example 1

$$A = \mathbb{R} \setminus \{0\}$$

$$R = \{(a, b) \in A \times A : (a, b) \in R \Leftrightarrow \frac{a}{b} \in \mathbb{Q}\}$$

Prove that this is an equivalence relation.

Reflexive: Let $a \in A$, since $a \neq 0$, $\frac{a}{a} = 1 \in \mathbb{Q}$ then $(a, a) \in R$.

Symmetric: Suppose $(a, b) \in R$, then $\frac{a}{b} \in \mathbb{Q}$ since $a \neq 0$. $\frac{b}{a} = (\frac{a}{b})^{-1} \in \mathbb{Q}$, so $(b, a) \in R$.

Transitive: Suppose $(a, b) \in R$ and $(b, c) \in R$, so $\frac{a}{b} \in \mathbb{Q}$ and $\frac{b}{c} \in \mathbb{Q}$. Thus, $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} \in \mathbb{Q}$, so $(a, c) \in R$.

This is an equivalence relation!

Example 2

$$A = \mathbb{Z} > 0$$

$$R = \{(a, b) \in R : a \mid b\}$$

Reflexive: **YES** If $a \in \mathbb{Z} > 0$, then $a \mid a$ since $a = a \cdot 1$.

Transitive: **YES** If $a \mid b$ and $b \mid c$, then $a \mid c$.

Symmetric: **NO** $2 \mid 6$, but $6 \nmid 2$.

This is not an equivalence relation!

Example 3

$$A = \mathbb{R}$$

$$R = \{(a, b) \in A \times A : |a - b| \leq 1\}$$

Reflexive: **YES** $\forall a \in A |a - a| = 0$, so $(a, a) \in R$.

Symmetric: **YES** If $(a, b) \in R$, then $|a - b| \leq 1$, so $|b - a| \leq 1$, so $(b, a) \in R$.

Transitive: **NO** $(3, 4) \in R$ since $|3 - 4| = 1$ and $(4, 5) \in R$ since $|4 - 5| = 1$, but $|3 - 5| = 2$, so $(3, 5) \notin R$

This is not an equivalence relation.

Equivalence Classes

$E_0 = [-1, 1]$ is the set of things related to zero.

$E_{\frac{1}{2}} = [-\frac{1}{2}, \frac{3}{2}]$ is the set of things related to $\frac{1}{2}$.

These overlap so they cannot be equivalence classes.

NOTE

For any set A , the relations

$$R_1 = \{(a, a) \in A \times A\} \text{ and}$$

$$R_2 = A \times A$$

are always equivalence relations.

R_1 is the relation where only elements are related to themselves and nothing else.

- Equivalence classes each contain exactly one element.

R_2 is the relation where everything in A is related to everything.

- Reflexive, symmetric, and transitive.
- Only one equivalence class contains the entire set of A .

Definitions:

Definition 1. A **relation** R on a set A is a subset of $A \times A$. (Often we write $a \sim b$ to mean $(a, b) \in R$.)

1. **Reflexive:** $(a, a) \in R$ for every $a \in A$.
2. **Symmetric:** If $(a, b) \in R$, then $(b, a) \in R$.
3. **Transitive:** If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Definition 2. A relation is an **equivalence relation** if it is reflexive, symmetric, and transitive.

1. If it is an equivalence relation, then it partitions A into equivalence classes of subsets of A that only contain related elements, and no elements between classes are related.

Theorems and Proofs: No theorems covered in this lecture.

Questions raised in class: All questions were fully settled in class.