*"Now, here, you see, it takes all the running you can do, to keep in the same place."*
Lewis Carroll, *Through the Looking-Glass*

**Written Assignment 5** **MATH 267: Introduction to Abstract Math**
Chapters 16–19 **Due: Monday, November 16, at the beginning of class**

**Name:** _____ **Collaborators:** _____

**Instructions.**

- **Write in complete sentences.** These are graded on the writing as well as on the mathematics. Say what you are assuming at the start and what you have concluded at the end, and do not make the reader guess which is which.
- **Collaboration is encouraged**, but write your final solutions in your own words and list your collaborators above. Generative AI may not be used on this assignment. See the course policies.
- Staple your pages.

Problems 1 and 2 can be done after the 10/28 class, Problems 3 and 4 after 11/2, and Problem 5 after 11/4. As before, explore before you prove: several problems suggest small cases to compute first. That work is not collected, but it is how you will find the proof.

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**Problem 1.** [10 points]

(a) Let $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = \sqrt{x}$. Prove that $f$ is strictly increasing. *(Suppose toward contradiction that $0 \leq x < y$ but $\sqrt{x} \geq \sqrt{y}$. Square both sides.)*

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be *any* strictly increasing bijection. Prove that $f^{-1}$ is also strictly increasing.

**Problem 2.** [10 points]

A student submits the following.

*"Theorem."* Let $f : A \to B$ and $g : B \to C$ be functions, and suppose $g \circ f$ is one-to-one. Then both $f$ and $g$ are one-to-one.

*"Proof."* Suppose $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$. Then $g(f(a_1)) = g(f(a_2))$, so $(g \circ f)(a_1) = (g \circ f)(a_2)$. Since $g \circ f$ is one-to-one, $a_1 = a_2$. So $f$ is one-to-one.

Now suppose $g(b_1) = g(b_2)$ for $b_1, b_2 \in B$. Since $f$ is onto, there exist $a_1, a_2 \in A$ with $f(a_1) = b_1$ and $f(a_2) = b_2$. Then $g(f(a_1)) = g(f(a_2))$, so by the same argument as above, $a_1 = a_2$. Therefore $b_1 = f(a_1) = f(a_2) = b_2$, so $g$ is one-to-one.

(a) The first paragraph is correct (it is Theorem 17.8(ii)). The second paragraph has exactly one unjustified step. Identify it, and explain why it cannot be justified from the given hypotheses.

(b) Is it actually true, for general functions $f$ and $g$ with $g \circ f$ one-to-one, that $g$ must be one-to-one? Prove it, or give specific finite sets $A$, $B$, $C$ and functions $f$, $g$ satisfying the hypothesis for which $g$ is *not* one-to-one.

**Problem 3.** [10 points]

Let $f : A \to B$ be a function, and let $C, D$ be nonempty subsets of $A$ with $C \cup D = A$ and $C \cap D = \emptyset$.

(a) Suppose $f|_C$ and $f|_D$ are both onto $B$. Must $f$ itself be one-to-one? Prove it, or give a specific counterexample using finite sets.

(b) Now suppose instead that $f$ is one-to-one. Prove that $f(C)$ and $f(D)$ are disjoint subsets of $B$.

(c) Suppose in addition to the hypothesis in (b) that $f$ is onto $B$. Prove that $\{f(C), f(D)\}$ is a partition of $B$.

**Problem 4.** [10 points]

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1 - x^2$.

(a) Find $f([-1,2])$ and prove your answer. *(Compute $f(-1)$, $f(0)$, and $f(2)$ first. Where on $[-1,2]$ is $f$ largest?)*

(b) Find $f^{-1}([-3,0])$ and prove your answer.

(c) Let $g : X \to Y$ be *any* function, and let $B_1, B_2 \subseteq Y$. Prove that $g^{-1}(B_1 \cup B_2) = g^{-1}(B_1) \cup g^{-1}(B_2)$.

**Problem 5.** [10 points]

Define a sequence by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

(a) Identify a set $X$, a function $f : X \to X$, and the starting value used in the recursion theorem to justify that this defines a sequence. *(It is not enough to say $X = \mathbb{R}$; find an $X$ on which $f$ actually maps into itself, and check it.)*

(b) Prove by induction that $1 \leq a_n < 2$ for every positive integer $n$.

(c) Prove by induction that $(a_n)$ is strictly increasing. *(You may use, without reproving it, that $\sqrt{\cdot}$ is strictly increasing on $[0,\infty)$ – this is Problem 1(a).)*

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*Total: 50 points.*