"Would you tell me, please, which way I ought to go from here?" "That depends a good deal on where you want to get to," said the Cat.

Lewis Carroll, *Alice's Adventures in Wonderland*

Written Assignment 4 — **MATH 267: Introduction to Abstract Math**

Chapters 11–13 and 15 — **Due: Wednesday, October 28, at the beginning of class**

Name: _____ **Collaborators:** _____

Instructions.

- **Write in complete sentences.** These are graded on the writing as well as on the mathematics. Say what you are assuming at the start and what you have concluded at the end, and do not make the reader guess which is which. When you prove that a relation is transitive, or that two sets in a collection are equal, name the elements you are working with before you use them.
- **Collaboration is encouraged**, but write your final solutions in your own words and list your collaborators above. Generative AI may not be used on this assignment. See the course policies.
- Staple your pages.

Problems 1 and 2 can be done after the 10/14 class, Problem 3 after 10/19, and Problems 4 and 5 after 10/21. As before, explore before you prove: several problems suggest small cases to compute first. That work is not collected, but it is how you will find the proof.

Problem 1. [10 points]

An integer m is a *perfect square* if $m = k^2$ for some integer k. (So 0, 1, 4, 9, ... are perfect squares.) On the positive integers $\mathbb{Z}^+$, define

$$a \sim b \quad \text{if and only if} \quad ab \text{ is a perfect square.}$$

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{Z}^+$. You may use without proof the following fact: if an integer m equals q^2 for some rational number q, then m is a perfect square. *(Before you start, list the elements of E_3 and of E_{12} that are at most 50. What do you notice?)*

(b) Now use the same rule on all of $\mathbb{Z}$: for $a, b \in \mathbb{Z}$, say $a \approx b$ if and only if ab is a perfect square. Decide which of the three properties (reflexive, symmetric, transitive) $\approx$ has. Prove the ones that hold and disprove the ones that do not, with specific integers. Then point to the step of your proof in (a) that no longer works.

Problem 2. [10 points]

Let X be a nonempty set, and let $\mathcal{A}$ and $\mathcal{B}$ be two partitions of X. Define a new collection of subsets of X by

$$\mathcal{C} = \{A \cap B : A \in \mathcal{A},\ B \in \mathcal{B},\ \text{and } A \cap B \neq \emptyset\}.$$

So the sets of $\mathcal{C}$ are the nonempty overlaps of a piece of $\mathcal{A}$ with a piece of $\mathcal{B}$.

(a) Take $X = \mathbb{Z}$, let $\mathcal{A}$ be the partition into even and odd integers, and let $\mathcal{B} = \{B_0, B_1, B_2\}$, where $B_i = \{3k + i : k \in \mathbb{Z}\}$. (You may use without proof that $\mathcal{A}$ and $\mathcal{B}$ are partitions of $\mathbb{Z}$.) List the sets in $\mathcal{C}$, with four elements of each. How many sets are there, and how would you describe them?

(b) Prove that $\mathcal{C}$ is a partition of X for any two partitions $\mathcal{A}$ and $\mathcal{B}$ of X.

Problem 3. [10 points]

(a) Let $S = \left\{(-1)^n + \frac{1}{n} : n \in \mathbb{Z}^+\right\}$. Find $\sup S$ and $\inf S$, say whether each one is a maximum or minimum of S, and prove all of your answers. *(Write out the elements for $n = 1, \ldots, 6$ first.)*

(b) Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, let $c > 0$, and let $cS = \{cs : s \in S\}$. Prove that cS is bounded above and that $\sup(cS) = c \sup S$.

Problem 4. [10 points]

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \dfrac{2x}{x^2+1}$. (The denominator is never zero, so this is a function.)

(a) Find $\mathrm{ran}(f)$ and prove your answer. *(Compute a few values, such as $f(0)$, $f(\pm 1)$, $f(2)$, and $f(\frac{1}{2})$, to make a guess. For one containment, compare $2|x|$ with x^2+1. For the other, start from y and solve $y = \frac{2x}{x^2+1}$ for x on scratch paper; then write the proof forward, as in Example 15.8.)*

(b) Now let $g : [1, \infty) \to \mathbb{R}$ be given by the same formula, $g(x) = \dfrac{2x}{x^2+1}$. Find $\mathrm{ran}(g)$ and prove your answer.

Problem 5. [10 points]

For an integer $n \geq 2$ and $x \in \mathbb{Z}$, let $[x]_n = \{y \in \mathbb{Z} : n \mid (x-y)\}$, the equivalence class of x under the relation "n divides $x - y$," and let $\mathbb{Z}_n = \{[x]_n : x \in \mathbb{Z}\}$ be the set of these classes. For example, $[1]_4 = [5]_4 = [-3]_4$. You may use without proof that $[x]_n = [x']_n$ if and only if $n \mid (x - x')$.

A rule such as $f([x]_4) = [x^2]_8$ is supposed to define a function from $\mathbb{Z}_4$ to $\mathbb{Z}_8$. The input is a class, but the rule is written in terms of one particular element x of that class. So before we may call f a function, we have to know that the output does not depend on which element we used to name the input. *(It may help to start by finding the remainders of x^2 and of x^3 upon division by 8 for $x = 0, 1, \ldots, 7$, and comparing x with $x + 4$.)*

(a) Prove that $f : \mathbb{Z}_4 \to \mathbb{Z}_8$, $f([x]_4) = [x^2]_8$, is a well-defined function. *(Say which of the two conditions in the definition of a function is actually in question here, and why the other one is automatic.)*

(b) Prove that $g([x]_4) = [x^3]_8$ does not define a function from $\mathbb{Z}_4$ to $\mathbb{Z}_8$.

Total: 50 points.