

“Begin at the beginning,” the King said gravely, “and go on till you come to the end: then stop.”

Lewis Carroll, *Alice’s Adventures in Wonderland*

Written Assignment 3

Chapters 7–10

MATH 267: Introduction to Abstract Math

Due: Monday, October 12, at the beginning of class

Name: _____

Collaborators: _____

Instructions.

- **Write in complete sentences.** These are graded on the writing as well as on the mathematics. Say what you are assuming at the start and what you have concluded at the end, and do not make the reader guess which is which. In an induction proof, say what the base step is, state the induction hypothesis, and point out where you use it.
- **You need nothing past Chapter 10.** You may cite the statements of Theorem 7.4 by number and the two principles of induction (Chapter 8). You may also use, without proof, the following fact about the real numbers: for every real number x there is a positive integer n with $n > x$.
- **Collaboration is encouraged**, but write your final solutions in your own words and list your collaborators above. Generative AI may not be used on this assignment. See the course policies.
- Staple your pages.

This assignment is due after Midterm Exam 1, but it covers the same material, so working it before Wednesday is good preparation. As before, some parts ask you to *explore* before you prove: compute small cases, look for the pattern, and only then commit to a claim.

Problem 1.

[10 points]

Set difference is not associative: where you put the parentheses in an expression like $A \setminus B \setminus C$ matters. This problem pins down exactly how much it matters. Throughout, A , B , and C are sets.

- Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.
- Give specific sets A , B , C for which $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$, and compute both sides to show they differ.
- Prove that $(A \setminus B) \setminus C = A \setminus (B \setminus C)$ if and only if $A \cap C = \emptyset$. (Part (a) is useful in both directions. For one direction, think about where an element of $A \cap C$ lands on each side.)

Problem 2.

[10 points]

- Prove by induction that $7 \mid (3^{2n+1} + 2^{n+2})$ for every natural number $n \geq 0$. (In the inductive step, write $3^{2n+3} = 9 \cdot 3^{2n+1}$ and $2^{n+3} = 2 \cdot 2^{n+2}$. The coefficients 9 and 2 do not match, so the expression is not simply a multiple of the old one. Write $9 = 2 + 7$ and see what that buys you.)
- Prove that $3^n > n^3$ for every integer $n \geq 4$. You may use without proof the fact that $(k+1)^3 < 3k^3$ for every integer $k \geq 4$. (Start the inductive step from $3^{k+1} = 3 \cdot 3^k$. Say exactly where you use the induction hypothesis and where you use this fact.)

Problem 3.

[10 points]

Define a sequence by

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-1} + 2a_{n-2} \quad \text{for } n \geq 3.$$

- Compute a_1 through a_8 . Then compute $a_n + a_{n+1}$ for $n = 1, \dots, 7$, and also $3a_n - 2^n$ for $n = 1, \dots, 8$. Describe the pattern you see in each list.
- Prove that $a_n + a_{n+1} = 2^n$ for every positive integer n . (Ordinary induction is enough here.)

- (c) Based on your second list in (a), conjecture a formula for a_n in terms of n , and prove it. You may use complete induction directly from the recursive rule, or ordinary induction together with part (b). Either way, be explicit about your base case or cases.

Problem 4.

[10 points]

For each positive integer n , let $A_n = [\frac{1}{n}, n]$. Write out A_1, A_2, A_3 . Find $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$, and prove both answers.

Problem 5.

[10 points]

Throughout, A, B, C, D are sets. Remember that the elements of a Cartesian product are ordered pairs; write them that way.

(a) Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(b) Now suppose $A \subseteq X$ and $B \subseteq Y$. A classmate claims that

$$(X \times Y) \setminus (A \times B) = (X \setminus A) \times (Y \setminus B).$$

Disprove the claim with a specific example (it may be a finite one). Then find a correct formula for $(X \times Y) \setminus (A \times B)$ as a union of two Cartesian products, and prove your formula.

Total: 50 points.