"Then Logic would take you by the throat, and force *you to it! Logic would tell you, 'You can't help yourself. No, you've* got *to accept Z!' So you've no choice, you see."*

Lewis Carroll, *What the Tortoise Said to Achilles*

Written Assignment 2 **MATH 267: Introduction to Abstract Math**

Chapters 5–6 **Due: Wednesday, September 23, at the beginning of class**

Name: ________________________ **Collaborators:** ________________________

Instructions.

- **Write in complete sentences.** These are graded on the writing as well as on the mathematics. Say what you are assuming at the start and what you have concluded at the end, and do not make the reader guess which is which.
- **You need nothing past Chapter 6:** the proof techniques of Chapter 5 and the element method for subsets and set equality from Chapter 6. Not unions, intersections, or complements (Chapter 7), even where a problem talks about two sets at once.
- **Collaboration is encouraged**, but write your final solutions in your own words and list your collaborators above. Generative AI may not be used on this assignment. See the course policies.
- Staple your pages.

As on Assignment 1, several parts ask you to *explore* before you prove anything: try small cases, notice a pattern, and only then commit to a claim. Start early. If you are stuck on one part for more than twenty minutes, that is a good thing to bring to office hours.

Problem 1. [10 points]

Let a and b be integers.

(a) Prove: if ab is even, then a is even or b is even. *(Look at the shape of the conclusion before you pick a technique.)*

(b) Prove: if a is even or b is even, then ab is even.

(c) Now combine the two ideas. Prove: if $a^2 + b^2$ is even, then $a + b$ is even. *(Neither a direct argument nor a single contrapositive sentence sees far enough here; look at what the four possible parities of the pair (a, b) do to $a^2 + b^2$.)*

Problem 2. [10 points]

Below are two attempted proofs. Both have a real gap: something asserted that has not actually been earned. For each one, say *precisely* what is missing or unjustified, naming the sentence and explaining why it is not allowed, and then repair the situation as described.

(a) **Claim.** There is no integer n with $n^2 + n = 5$.

Attempted proof. Suppose, for contradiction, that $n^2 + n = 5$ for some integer n. By the quadratic formula, $n = \frac{-1\pm\sqrt{21}}{2}$, which is irrational. But n was assumed to be an integer, and every integer is rational. This is a contradiction, so no such n exists. ■

After you identify the gap, give a *different* proof of the same claim, one that does not have this problem. *(Hint: rewrite $n^2 + n$ as $n(n + 1)$, a product of consecutive integers, and think about which pairs of integers multiply to 5.)*

(b) **Claim.** For every integer n that is not a multiple of 3, n^2 leaves remainder 1 when divided by 3.

Attempted proof. Since n is not a multiple of 3, write $n = 3k + 1$ for some integer k. Then

$$n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1,$$

and $3k^2 + 2k$ is an integer, so n^2 leaves remainder 1 when divided by 3. ■

After you identify the gap, supply what is missing and complete the proof.

Problem 3. [10 points]

In lecture we proved that $3 \mid (n^3 - n)$ for every integer n. This problem pushes that result further.

(a) Prove: for every integer n, $2 \mid (n^3 - n)$. *(Hint: factor $n^3 - n = (n-1)n(n+1)$, then argue by cases on the parity of n.)*

(b) You may use, without proof, the fact that if $2 \mid x$ and $3 \mid x$, then $6 \mid x$. Using this fact, part (a), and the lecture's theorem that $3 \mid (n^3 - n)$, prove that $6 \mid (n^3 - n)$ for every integer n.

(c) Explore: is it also true that $24 \mid (n^3 - n)$ for every integer n? Compute $n^3 - n$ for $n = 0, 1, 2, 3, 4, 5$ and check each value against 24. State a conjecture (true for all n, or false) and, if it is false, give a specific counterexample. No proof is required beyond your computations and a clearly stated conclusion.

Problem 4. [10 points]

Recall from Assignment 1 the sets $D_n = \{x \in \mathbb{Z} : n \mid x\}$. There you conjectured, and were only asked to *test*, that $\{x \in \mathbb{Z} : x \in D_4 \text{ and } x \in D_6\} = D_{12}$. Now that we have the element method, you can actually prove statements like this.

(a) Prove: $\{x \in \mathbb{Z} : x \in D_4 \text{ and } x \in D_6\} = D_{12}$. *(For the harder direction, let $x = 4a = 6b$, simplify to $2a = 3b$, and reuse the parity idea from Problem 1.)*

(b) Prove: $\{x \in \mathbb{Z} : x \in D_9 \text{ and } x \in D_{15}\} = D_{45}$. *(Same strategy as (a), but with cases on the remainder of b upon division by 3.)*

(c) Based on (a) and (b), state a conjecture for what single number k satisfies $\{x \in \mathbb{Z} : x \in D_m \text{ and } x \in D_n\} = D_k$ in general, in terms of m and n. Test your conjecture on the pair $m = 8$, $n = 12$: list a few small elements of $\{x \in \mathbb{Z} : x \in D_8 \text{ and } x \in D_{12}\}$ and check them against your predicted D_k. No proof is required for this part.

Problem 5. [10 points]

Let

$$O = \{x \in \mathbb{Z} : x = 2n + 1 \text{ for some } n \in \mathbb{Z}\}, \qquad T = \{x \in \mathbb{Z} : x = 3n + 1 \text{ for some } n \in \mathbb{Z}\}.$$

So O is the odd integers, and T is the integers that leave remainder 1 when divided by 3.

(a) List the six smallest nonnegative elements of T. Based on this list and your knowledge of O, do you expect $O \subseteq T$, $T \subseteq O$, both, or neither? You do not need to justify this part; it is a prediction to check against (b) and (c).

(b) Prove or disprove: $O \subseteq T$.

(c) Prove or disprove: $T \subseteq O$.

(d) Parts (b) and (c) show that neither of O, T contains the other, but their overlap is still an interesting set. Let

$$U = \{x \in \mathbb{Z} : x \in O \text{ and } x \in T\}.$$

Prove that $U = \{x \in \mathbb{Z} : x = 6n+1 \text{ for some } n \in \mathbb{Z}\}$. *(For the harder containment, an element of U satisfies $x = 2a + 1 = 3b + 1$ for integers a, b; simplify to a relationship between a and b, and reuse the parity idea from Problem 4.)*

Total: 50 points.