"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."
Lewis Carroll, *Alice's Adventures in Wonderland / Through the Looking-Glass*

Written Assignment 1 — **MATH 267: Introduction to Abstract Math**
Chapters 2–4 — **Due: Wednesday, September 9, at the beginning of class**

Name: ________________________ **Collaborators:** ________________________

Instructions.

- **Write in complete sentences.** These are graded on the writing as well as on the mathematics. An argument that is correct but unreadable will not receive full credit. Say what you are assuming at the start and what you have concluded at the end, and do not make the reader guess which is which.
- **You need nothing past Chapter 4.** Every problem here can be done with the logic, contrapositive, set notation, and quantifiers we have already covered, together with the definition of *divides* from Chapter 1. You are welcome to use a direct argument or an argument by contrapositive, both of which we have done in class; you do not need any technique from Chapter 5.
- **Collaboration is encouraged**, but write your final solutions in your own words and list your collaborators above. Generative AI may not be used on this assignment. See the course policies.
- Staple your pages. Late work is accepted for reduced credit up to one week past the due date.

These problems are meant to be harder than the practice problems, and several of them ask you to *explore* before you prove anything: try small cases, notice a pattern, and only then commit to a claim. Start early. If you are stuck on one part for more than twenty minutes, that is a good thing to bring to office hours.

Problem 1. [10 points]

In 1913 Henry Sheffer noticed something surprising about the connectives in Chapter 2: you do not need all of them. Define a new connective $\uparrow$, called the *Sheffer stroke*, by declaring that $P \uparrow Q$ is true exactly when at least one of P, Q is false. (Read it as "not both." If you have seen a NAND gate in a computing course, this is that.)

(a) Write the truth table for $P \uparrow Q$, and then write the truth table for $P \uparrow P$. What familiar statement form is $P \uparrow P$ equivalent to?

(b) Find a statement form built from P and Q using the connective $\uparrow$ *and no other connective* that is equivalent to $P \wedge Q$. Verify your answer with a truth table.

(c) Now the other direction. Explain carefully why there is *no* statement form built from P using only $\wedge$ and $\vee$ that is equivalent to $\neg P$. *(Hint: think about the row of the truth table in which every letter is true. What is the truth value of any form built only from $\wedge$ and $\vee$ in that row?)* A clear paragraph explaining the key observation is what is wanted here; making it fully airtight requires induction, which is Chapter 18.

Problem 2. [10 points]

Below are two attempted proofs. Both are wrong. For each one, your job is to say *precisely* where it goes wrong (naming the sentence that fails and explaining why it is not allowed) and then to repair the situation as described.

(a) **Claim.** Let x be an integer. If x^3 is even, then x is even.

Attempted proof. Suppose x^3 is even. Then $x^3 = 2k$ for some integer k. Since x is even, we may write $x = 2m$ for some integer m. Then $x^3 = (2m)^3 = 8m^3 = 2(4m^3)$, and $4m^3$ is an integer, so x^3 is even, as desired. ■

Identify the error. Then state precisely which statement, if any, this argument *does* correctly prove. Finally, give a correct proof of the original claim. *(The claim is true. You have seen the technique that works; Chapter 3 is the relevant chapter.)*

(b) **Claim.** Let x be a real number. If $x^2 > 4$, then $x > 2$.

Attempted proof. Suppose $x^2 > 4$. Then $x^2 - 4 > 0$, so $(x-2)(x+2) > 0$. A product of two numbers is positive, so $x - 2 > 0$, and therefore $x > 2$. ■

Identify the error. Then decide whether the *claim* is true or false and say how you know. If it is false, state a corrected version that is true, and prove it. *(Note that these are two different questions: an argument can be broken even when its conclusion is true, and here you should be careful to say which situation you are in.)*

Problem 3. [10 points]

Every implication $P \to Q$ comes with three relatives:

$$\text{converse } Q \to P, \qquad \text{inverse } \neg P \to \neg Q, \qquad \text{contrapositive } \neg Q \to \neg P.$$

(a) Show by truth table that the converse and the inverse are logically equivalent. Then explain the same fact in one sentence *without* a truth table, using only what you know about contrapositives.

(b) Using part (a) and Theorem 3.2 (an implication is equivalent to its contrapositive), explain why the four sentences above always fall into exactly two groups, with the members of each group sharing a truth value. What is the largest number of distinct truth values that can occur among the four?

(c) Now the exploration. For each of the four combinations below, give an implication about integers, in English, that realizes it, and justify each truth value briefly. Use a different mathematical idea for each of the four if you can.

	statement	*converse*
(i)	true	true
(ii)	true	false
(iii)	false	true
(iv)	false	false

Problem 4. [10 points]

Let $A = \{1, 2, 3, 4\}$ and let $p(x, y)$ be a property of pairs of elements of A. The table below records p completely: the entry in row x and column y is ✓ if $p(x, y)$ is true, and $\cdot$ if $p(x, y)$ is false. Throughout this problem, all quantified variables range over A.

$p(x,y)$	$y=1$	$y=2$	$y=3$	$y=4$
$x=1$	✓	$\cdot$	$\cdot$	$\cdot$
$x=2$	$\cdot$	✓	✓	$\cdot$
$x=3$	$\cdot$	✓	$\cdot$	✓
$x=4$	$\cdot$	✓	$\cdot$	✓

(a) Decide whether each of the following is true or false. In each case justify your answer in one sentence that refers to the rows or columns of the table: part of the point of this problem is to learn to *see* a quantifier.

(i) $\forall x,\ \exists y : p(x, y)$

(ii) $\exists y : \forall x,\ p(x,y)$
(iii) $\forall x,\ \forall y,\ \big(p(x,y) \to p(y,x)\big)$

(b) Write the negation of (i) and the negation of (iii) in symbols. Then say, in a sentence about the table, what each negation asserts: for instance, one of them says that some row is empty.

(c) Design your own 4×4 table for which $\forall x,\ \exists y : p(x,y)$ is true but $\forall y,\ \exists x : p(x,y)$ is false. Draw the table and explain in one sentence why it works.

(d) Prove that for *every* such table,

$$\big(\exists y : \forall x,\ p(x,y)\big) \ \longrightarrow\ \big(\forall x,\ \exists y : p(x,y)\big).$$

So the order of the quantifiers matters, but not symmetrically: one of the two orders always implies the other. Then explain why part (a) shows that the reverse implication is false in general.

Problem 5. [10 points]

For a positive integer n, define

$$D_n = \{x \in \mathbb{Z} : n \mid x\}.$$

(a) Describe D_6 in words, and list its five smallest nonnegative elements. What set is D_1? Is $0 \in D_n$ for every n? Justify your answer from the definition of "divides," not from intuition.

(b) Explore. For each pair (m, n) with $1 \le m \le 6$ and $1 \le n \le 6$, decide whether every element of D_m is an element of D_n, and record your answers in a 6×6 table. *(You do not need to justify all thirty-six entries; do enough of them carefully that you trust the table.)*

(c) Look at your table and state a conjecture: a simple condition on m and n that holds exactly when every element of D_m is an element of D_n. Then prove your conjecture. *(It is an "if and only if," so there are two things to prove. For one direction, the observation that m itself is an element of D_m is the whole trick.)*

(d) One more experiment. Describe the set $\{x \in \mathbb{Z} : x \in D_4 \text{ and } x \in D_6\}$ in words, and identify which single set D_k it is equal to. Then make a guess: for general m and n, which D_k do you get? Test your guess on one more pair of your own choosing and report what happened. *(No proof required for this part: I want the guess and the test.)*

Total: 50 points.