Homework Assignment 4 **Math 369 - Introduction to Abstract Algebra**

"In mathematics you don't understand things. You just get used to them." — John von Neumann

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Instructions: Show all work clearly. Justify your answers with complete explanations. Late assignments will not be accepted without prior arrangement.

Problem 1. Let $n \geq 2$ be an integer.

(a) Prove that if $\phi : \mathbb{Z} \to \mathbb{Z}_n$ is a ring homomorphism, then $\phi(k) = [k]_n$ for all $k \in \mathbb{Z}$. (That is, the only ring homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_n$ is the natural projection.)

(b) More generally, let $\phi : \mathbb{Z} \to R$ be a ring homomorphism where R is any commutative ring. Show that ϕ is completely determined by $\operatorname{char}(R)$.

Problem 2. Let F_1 and F_2 be fields. The direct sum $F_1 \oplus F_2$ consists of all ordered pairs (a, b) with $a \in F_1$ and $b \in F_2$, under componentwise operations:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \quad \text{and} \quad (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2)$$

(a) Show that $F_1 \oplus F_2$ is a commutative ring with identity.

(b) Prove that $F_1 \oplus F_2$ is not a field.

Problem 3. Let $R = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers, and let $I = \langle 5 \rangle$ be the principal ideal generated by 5.

(a) Show that $5 = (1 + 2i)(1 - 2i)$ in $\mathbb{Z}[i]$.

(b) Use part (a) to prove that I is not a prime ideal.

(c) Determine whether I is a maximal ideal in $\mathbb{Z}[i]$.

Problem 4. Let R be an integral domain of characteristic $p > 0$. The *Frobenius homomorphism* is the map $\phi : R \to R$ defined by $\phi(a) = a^p$.

(a) Prove that ϕ is a ring homomorphism. (Hint: You will need the Binomial Theorem and the fact that p divides $\binom{p}{k}$ for $0 < k < p$.)

(b) Show that if R is a finite field, then ϕ is an isomorphism.

Problem 5. Let R be a commutative ring and let I, J be ideals of R with $I + J = R$ (meaning there exist $a \in I$ and $b \in J$ such that $a + b = 1$).

(a) Prove that for any $r \in R$, there exist $x \in I$ and $y \in J$ such that $r = x + y$.

(b) Define $\phi : R \to R/I \times R/J$ by $\phi(r) = (r + I, r + J)$. Prove that ϕ is a ring homomorphism with $\ker(\phi) = I \cap J$.

(c) Use the Fundamental Homomorphism Theorem to conclude that $R/(I \cap J) \cong R/I \times R/J$.