Homework Assignment 3 **Math 369 - Introduction to Abstract Algebra**

"Rigour is to the mathematician what morality is to men." — André Weil

Instructions: Show all work clearly. Justify your answers with complete explanations. Late assignments will not be accepted without prior arrangement.

Problem 1. Let G be an abelian group. Define

$$H = \{x^2 : x \in G\} \quad \text{and} \quad K = \{x \in G : x^2 = e\}.$$

Prove that $H \cong G/K$.

Problem 2. Let G and H be groups. Suppose J is a normal subgroup of G and K is a normal subgroup of H. Prove that

$$(G \times H)/(J \times K) \cong (G/J) \times (H/K).$$

Problem 3. Consider the factor group $\mathbb{R}/\mathbb{Z}$ where $\mathbb{R}$ is the additive group of real numbers and $\mathbb{Z}$ is the subgroup of integers.

(a) Show that $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$.
(b) Prove that every coset $r + \mathbb{Z}$ has a unique representative in the interval $[0, 1)$.
(c) Consider the unit circle group $C = \{$ angles in $[0, 2\pi)\}$ under addition modulo 2π (where two angles are equivalent if they differ by a multiple of 2π). Find an isomorphism $\phi : \mathbb{R}/\mathbb{Z} \to C$ and prove that it is indeed an isomorphism.
(d) Find all elements of order 2 or 3 in $\mathbb{R}/\mathbb{Z}$.
(e) Show that $[\pi] = \pi + \mathbb{Z}$ has infinite order in $\mathbb{R}/\mathbb{Z}$.

Problem 4. Let $p(x) = x^2 - 3 \in \mathbb{Q}[x]$ and let $E = \mathbb{Q}[x]/\langle x^2 - 3\rangle$.

(a) Show that $p(x)$ is irreducible over $\mathbb{Q}$ and conclude that E is a field.
(b) Compute the following products in E, expressing your answers in the form $[a + bx]$ where $a, b \in \mathbb{Q}$:
  (i) $[2 + x] \cdot [1 - x]$
  (ii) $[1 + x]^2$
  (iii) $[x]^3$
(c) Find the multiplicative inverse of $[2 + x]$ in E.
(d) Show that $E \cong \mathbb{Q}(\sqrt{3})$ by describing an explicit isomorphism.

Problem 5. Consider $p(x) = x^2 + x + 2 \in \mathbb{Z}_3[x]$ and let $F = \mathbb{Z}_3[x]/\langle x^2 + x + 2\rangle$.

(a) Verify that $p(x)$ is irreducible over $\mathbb{Z}_3$ and determine $|F|$.
(b) List all elements of F in the form $[a + bx]$ where $a, b \in \mathbb{Z}_3$.
(c) Compute $[x]^3$, $[1 + x]^2$, and $[2 + x] \cdot [1 + 2x]$ in F.
(d) Show that $F^\times$ is cyclic by finding a generator, where $F^\times = F \setminus \{[0]\}$ is the multiplicative group of nonzero elements.

Problem 6. Use the Euclidean algorithm to find the greatest common divisor of

$$f(x) = x^7 + 3x^5 + 8x^3 + x^2 + 24x + 3 \quad \text{and} \quad g(x) = x^4 + x^3 + 4x^2 + 3x + 3$$

in $\mathbb{Q}[x]$. Show all steps of your computation.