Homework Assignment 2 **Math 369 - Introduction to Abstract Algebra**

"In mathematics you don't understand things. You just get used to them." — John von Neumann

Instructions: Show all work clearly. Justify your answers with complete explanations. Late assignments will not be accepted without prior arrangement.

Problem 1. Let $G = \{e, a, b, c, d, f\}$ be a group with the following Cayley table:

$\cdot$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	f	c	d
b	b	e	a	d	f	c
c	c	d	f	e	a	b
d	d	f	c	b	e	a
f	f	c	d	a	b	e

(a) Find the order of each element in G.
(b) Find all subgroups of G and determine which ones are cyclic.
(c) Show that $G \cong S_3$ by finding an explicit isomorphism $\phi : G \to S_3$.
(d) Use your isomorphism to identify which elements of G correspond to transpositions in S_3.

Problem 2. Consider the group $(\mathbb{Z}_{18}, +)$.

(a) Find all generators of $\mathbb{Z}_{18}$. Explain your reasoning.
(b) For each divisor d of 18, find the unique subgroup of order d in $\mathbb{Z}_{18}$.
(c) Draw the subgroup lattice for $\mathbb{Z}_{18}$, showing all subgroups and their inclusion relationships.
(d) Find all elements of order 6 in $\mathbb{Z}_{18}$.

Problem 3. Prove that every group of order 4 is isomorphic to $\mathbb{Z}_4$ or $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $V_4 = \{e, a, b, c\}$ with $a^2 = b^2 = c^2 = e$ and $ab = c$, $ac = b$, $bc = a$.

Problem 4. Suppose G is a finite group of odd order. Show that $f : G \to G$, $f(x) = x^2$ is an isomorphism if and only if G is abelian.

Problem 5. Find an isomorphism from the group $(\mathbb{R}\backslash\{0\}, \cdot)$ where $\cdot$ is multiplication to the group $(\mathbb{R}\backslash\{0\}, *)$ where $a * b = \frac{ab}{\pi}$.

Problem 6. For each statement below, either prove it is true or provide a counterexample.

(a) If G is an infinite abelian group, then G is cyclic.
(b) If $\phi : G \to H$ is an isomorphism and G is not cyclic, then H is not cyclic.
(c) If G is an abelian group of order 8, then G is cyclic.
(d) If G and H are nonabelian groups with $|G| = |H|$, then $G \cong H$.