Homework Assignment 1 **Math 369 - Introduction to Abstract Algebra**

"The theory of groups is a branch of mathematics in which one does something to something and then compares the result with the result obtained from doing the same thing to something else, or something else to the same thing." — James R. Newman

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Instructions: Show all work clearly. Justify your answers with complete explanations. Late assignments will not be accepted without prior arrangement.

Problem 1. Let $\alpha = (1\,5\,3\,8)(2\,6)(4\,7)$ and $\beta = (1\,2\,4\,3)(5\,7\,6)$ be permutations in S_8.

(a) Write α in two-line notation.
(b) Compute $\alpha\beta$ and express your answer in both two-line notation and cycle notation.
(c) Compute $\beta\alpha$ and express your answer in both two-line notation and cycle notation.
(d) Find α^{-1} and express it in cycle notation.
(e) Find the orders of α, β, $\alpha\beta$, and $\beta\alpha$.

Problem 2. We proved in class that every permutation can be written as a product of transpositions.

(a) Show by example that this decomposition into transpositions is not unique. Specifically, write the permutation $(1\,2\,3\,4)$ as a product of transpositions in two different ways.
(b) A permutation is called **even** if it can be written as a product of an even number of transpositions, and **odd** if it can be written as a product of an odd number of transpositions. Note that a permutation cannot be both even and odd. Show that the set of all even permutations in S_3 forms a group under composition.

Problem 3. For each of the following, either prove the statement is true or provide a counterexample to show it is false.

(a) If σ and τ are permutations with the same order, then the cycles in the cycle structures of σ and τ have the same lengths.
(b) If σ and τ are permutations such that $\sigma\tau = \tau\sigma$, then σ and τ are disjoint.
(c) If σ is a permutation of order n, then σ^{-1} also has order n.

Problem 4. Consider the following algebraic structures with their operation tables. For each structure, determine whether it satisfies the properties listed below. Justify each answer.

Properties to check:

- **Magma:** Closed under the operation
- **Semigroup:** Magma with associativity
- **Quasigroup:** Magma where each row and column of the operation table contains each element exactly once
- **Monoid:** Semigroup with identity element
- **Loop:** Quasigroup with identity element
- **Group:** Monoid where every element has an inverse

Structure A: Set $\{a, b, c\}$ with operation $*$:

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Structure B: Set $\{a, b, c\}$ with operation $\circ$:

$\circ$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

Structure C: Set $\{a, b, c\}$ with operation $\diamond$:

$\diamond$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

Problem 5. Find examples of the following algebraic structures on a set $\{a, b, \ldots\}$ (different from those above) by providing their operation tables. You must construct **different** structures for each part. Try to make your examples as small as possible.

(a) A magma that is not a semigroup (fails associativity).
(b) A monoid that is not a group (lacks inverses for some elements).
(c) A loop that is not a group (fails associativity). *Hint: See the extended notes on the course website for more information about loops.*

For each construction, explicitly verify that your structure satisfies the required properties and fails the excluded ones.

Problem 6. Let $(G, *)$ be a group.

(a) Prove that if $a * b = a * c$ for elements $a, b, c \in G$, then $b = c$. (This is called the left cancellation property.)
(b) Prove that the equation $a * x = b$ has a unique solution $x \in G$ for any given elements $a, b \in G$.