Math 675
Discussion Questions
Chapter #6, Part 1

Reading: Chapter 6, Section 6.4

66. Find the leading behavior of
$$\int_0^{\pi/2} e^{-x \tan t} dt$$
as $x \to \infty$.

67. Find the leading behavior of
$$\int_0^{\infty} e^{-x \sinh^2 t} dt$$
as $x \to \infty$.

68. Find the leading behavior of
$$\int_{-\pi/2}^{\pi/2} (t+2)e^{-x \cos t} dt$$
as $x \to \infty$.

69. Find the leading behavior of
$$\int_{-1}^{1} e^{-x \sin^2 t} dt$$
as $x \to \infty$.

70. Consider the integral
$$\int_0^{\infty} f(t)e^{x\phi(t)} dt,$$
and assume
• $f(t)$ and $\phi(t)$ are continuous;
• $f(t) \geq 0$ with $f(t)$ not identically zero on any interval $[0,a]$;
• $\phi(t)$ has a single unique maximum at $t = 0$;
• $\int_0^{\infty} f(t)e^{\phi(t)} dt < \infty$.

(a) For any $a > 0$, define $\phi_*(a) = \min\{\phi(t) : 0 \leq t \leq a\}$. Prove
$$\int_0^{\infty} f(t)e^{x\phi(t)} dt \geq e^{x\phi_*(a)} \int_0^{a} f(t) dt.$$

(b) For any $b > 0$, define $\phi^*(b) = \max\{\phi(t) : t \geq b\}$. Prove that
$$\int_b^{\infty} f(t)e^{x\phi(t)} dt \leq e^{(x-1)\phi^*(b)} \int_0^{\infty} f(t)e^{\phi(t)} dt$$
for $x \geq 1$.

(c) For any $\epsilon > 0$, prove
$$\int_0^{\infty} f(t)e^{x\phi(t)} dt \sim \int_0^{\epsilon} f(t)e^{x\phi(t)} dt$$
as $x \to \infty$. 

71. Suppose that

- \( f(t), g(t), \) and \( \phi(t) \) are continuous;
- \( f(t) \geq 0 \) and \( g(t) \geq 0 \), with each not identically zero on any interval \([0, a]\);
- \( \phi(t) \) has a single unique maximum at \( t = 0 \);
- \( \int_0^\infty f(t)e^{\phi(t)} \, dt < \infty \) and \( \int_0^\infty f(t)e^{\phi(t)} \, dt < \infty \);
- \( f(t) \sim g(t) \) as \( t \to 0^+ \).

Prove

\[
\int_0^\infty f(t)e^{\phi(t)} \, dt \sim \int_0^\infty g(t)e^{\phi(t)} \, dt
\]
as \( x \to \infty \).

72. Read Example 7, p. 269 carefully. Consider the example

\[
\int_\epsilon^0 t e^{-x \sinh^4 t} \, dt.
\]

(a) Use the Mean Value Theorem or otherwise to prove that there is a \( \delta > 0 \) so that \( 0 \leq \sinh t \leq (1 + \delta) t \) for all nonnegative \( t \) sufficiently small.

(b) Use Taylor’s Theorem with remainder or otherwise to prove that there is a \( \delta > 0 \) so that \( t^3/6 \leq \sinh t - t \leq (1 + \delta) t^3/6 \) for all nonnegative \( t \) sufficiently small.

(c) Conclude that

\[
|\sinh^4 t - t^4| \leq \frac{2}{3}(1 + \delta)^4 t^6
\]

for all \( t \) sufficiently small.

(d) Let \( \alpha > 1/6 \). Prove

\[
\lim_{x \to \infty} \frac{\int_0^x t e^{-x \sinh^4 t} \, dt}{\int_0^x t e^{-t^4} \, dt} = 1
\]

provided \( \epsilon \) is sufficiently small.

(e) Show that \( \sinh t \geq t \) for all \( t \).

(f) Prove that

\[
\int_{x^\alpha}^x t e^{-x \sinh^4 t} \, dt \leq e^2 \exp(-x^{1-4\alpha}).
\]