Reading: Chapter 3, Sections 3.4

35. Prove each of the following:
   (a) \( x \ll 1/x \) as \( x \to 0^+ \).
   (b) \( 1/x \ll x \) as \( x \to \infty \).
   (c) \( x^2 \ll -25 \) as \( x \to 0 \).

36. Prove or disprove: If \( f(x) \ll g(x) \) as \( x \to x_0 \) and \( g(x) \ll h(x) \) as \( x \to x_0 \), then \( f(x) \ll h(x) \) as \( x \to x_0 \).

37. Prove or disprove: If \( f_1(x) \ll g(x) \) as \( x \to x_0 \) and \( f_2(x) \ll g(x) \) as \( x \to x_0 \), then \( (f_1(x)+f_2(x)) \ll g \) as \( x \to x_0 \).

38. Prove or disprove: If \( f(x) \ll g(x) \) as \( x \to x_0 \), then \( f'(x) \ll g'(x) \) as \( x \to x_0 \).

39. Prove or disprove: If \( f(x) \ll g(x) \) as \( x \to x_0 \), then \( \int_{x_0}^{x} f(t) \, dt \ll \int_{x_0}^{x} g(t) \, dt \).

40. Prove or disprove: \( x + e^x \sim e^x \) as \( x \to \infty \).

41. Prove or disprove: There is no function \( f(x) \) with \( f(x) \sim 0 \) as \( x \to x_0 \).

42. Let \( f(x) = x - \frac{1}{3} x^3 + \frac{1}{120} x^5 \). Prove
   (a) \( f(x) \sim x \) as \( x \to 0 \).
   (b) \( f(x) \sim x - \frac{1}{3} x^3 \) as \( x \to 0 \).
   (c) \( f(x) \sim x - \frac{1}{3} x^3 + 17 x^5 \) as \( x \to 0 \).

43. Prove or disprove: If \( f \sim g \) as \( x \to x_0 \) and \( \alpha \in \mathbb{C} \), then \( \alpha f \sim \alpha g \) as \( x \to x_0 \).

44. Prove or disprove: If \( f \sim \phi \) and \( g \sim \psi \) as \( x \to x_0 \), then \( (f + g) \sim (\phi + \psi) \) as \( x \to x_0 \).

45. Prove or disprove: If \( f \sim g \) and \( g \sim h \) as \( x \to x_0 \), then \( f \sim h \) as \( x \to x_0 \).

46. Prove or disprove: If \( f^2 \sim g^2 \) as \( x \to x_0 \), then \( f \sim g \) as \( x \to x_0 \).

47. Prove or disprove: If \( f = g + h \) and \( h \ll g \) as \( x \to x_0 \), then \( f \sim g \) as \( x \to x_0 \).

48. Consider the equation \( x^4 y'' = y \).
   (a) Show that this equation has an irregular singular point at the origin.
   (b) Show that this equation has no solution in Frobenius form.
   (c) Make the assumption \( y = e^{S(x)} \) and obtain a differential equation for \( S(x) \). Compare your result to (3.4.8).
   (d) By assuming that \( S'' \ll (S')^2 \) as \( x \to 0^+ \), obtain two asymptotic differential relations for \( S(x) \). Compare your result to (3.4.10).
   (e) Find a solution of each asymptotic differential relation.
Let $\sigma(x)$ be a solution of one of the asymptotic differential relations in (48d). Let $C(x)$ be any other function with $C' \ll \sigma'$ as $x \to 0+$. Prove that $\sigma(x) + C(x)$ also solves (48d).

What is the controlling factor for each solution of $x^4y'' = y$ as $x \to 0+$?

For the solution $S(x) = \sigma(x)$ found in (48e), verify that the assumption $S'' \ll (S')^2$ as $x \to 0+$ made in (48d) obtains.

Write $S(x) = \sigma(x) + C(x)$, and substitute this into (48c) to obtain a differential equation for $C(x)$.

Make the assumption $C'' \ll \sigma'$ as $x \to 0+$ and convert the equation in (48i) to a differential asymptotic relation.

Assume that $C''$ is much smaller than the nonhomogeneous term as $x \to 0+$ in (48j) to reduce the differential asymptotic relation to one with one term on each side. Solve it.

Verify that the solution obtained in (48k) satisfies all of the smallness assumptions you made.

What is the leading behavior of the solutions to $x^4y'' = y$ as $x \to 0+$?

49. Now consider the equation $x^4y''' = y$.

Make the assumption $y = e^{S(x)}$ and obtain a third order differential equation for $S(x)$.

Use the method of dominant balance (pp. 83–84) to solve an appropriate differential asymptotic relation. Call the non-oscillatory solution $\sigma(x)$.

Look for a correction in the form $S(x) = \sigma(x) + C(x)$; substitute this form into the equation (49a) to obtain a differential equation for $C(x)$. A computer algebra system like Mathematica is useful for the algebra.

Use the method of dominant balance on the equation in (49c); call the resulting function $\gamma(x)$.

Write $S(x) = \sigma(x) + \gamma(x) + D(x)$ for another correction $D(x)$. Substitute this form into the equation (49a) to obtain a differential equation for $D(x)$. The algebra is routine but painful. I did recommend a computer algebra system, right?

Apply the method of dominant balance to the equation in (49e).

What is the leading behavior of the non-oscillatory solution to $x^4y''' = y$?

Look for a correction to the leading order behavior by multiplying it by $w(x) = 1 + \epsilon(x)$, following (3.4.17). Obtain a differential equation for $\epsilon(x)$. You do realize that the bit about a computer algebra system was meant seriously, right?

Apply the method of dominant balance to the equation in (49h).

Note that this process can be continued to obtain additional terms in the expansion of $w(x)$. 