Math 674
Discussion Questions
Chapter #3, Part 1

Reading: Chapter 3, Sections 3.1-3.3.

23. (§1.7). Consider the equation
\[ x^2 y'' + 3xy' + 2y = x^{-4}y^{-3}. \]

(a) Let \( \xi = ax \) and \( \eta = a^{-1}y \), and show that the equation is unchanged.
(b) Write \( y = u/x \) and reduce the equation to one that is equidimensional in \( x \).

24. (§3.1). For each of the following, determine if 0 is an ordinary point, a regular singular point, or an irregular singular point.
(a) \( x^7 y^{(4)} = y' \).
(b) \( x^3 y''' = y \).
(c) \( y''' = x^3 y \).
(d) \( x^2 y'' = e^{1/x}y \).
(e) \( (\tan x)y' = y \).

25. For each of the equations in 24, classify the point at infinity as an ordinary point, a regular singular point, or an irregular singular point.

26. Consider the equation
\[ (\sin x)y' - y = 0. \]

(a) Find all of the singular points of the equation, and classify them as regular or irregular.
(b) The solution of this equation is analytic at the origin. Justify this statement.
(c) Find a lower bound for the radius of convergence of the power series at the origin of the solution.
(d) Use techniques from first order linear equations to solve the problem. [Hint: The half-angle formula for \( \tan x/2 \) is particularly convenient.
(e) Show that the solution so determined is analytic, and determine its radius of convergence.

27. (§3.2). Consider the equation
\[ y'' - 2xy' + 8y = 0. \]

(a) Solve the equation subject to the conditions \( y(0) = 4, \ y'(0) = 0 \).
(b) Solve the equation subject to the conditions \( y(0) = 0, \ y'(0) = 4 \).

28. (§3.3). Consider Bessel’s equation of order \( \nu \)
\[ x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \]  \( (1) \)

(a) Is the origin an ordinary point, a regular singular point, or an irregular singular point?
(b) Write (1) in the form (3.3.2), and identify the functions \( p(x) \) and \( q(x) \). Verify that \( p(x) \) and \( q(x) \) are analytic at the origin.
(c) Make the Forbenius ansatz

\[ y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \]

for \( a_0 \neq 0 \). Write out the indicial equation and the recursion relations.

(d) Compare your indicial equation to (3.3.4), using \( p(x) \) and \( q(x) \) from (28b). Verify they match.

29. Find a solution of Bessel’s equation corresponding to the larger root of the indicial equation. Compare your result to p.572, #2.

30. Consider Bessel’s equation and suppose that \( \nu \) and \( 2\nu \) are not integers. Find a second solution of Bessel’s equation using the smaller root of the indicial equation, following Case I, p. 72. Be sure to identify the point(s) in your argument where you use the assumption that \( 2\nu \) is not an integer.

31. Consider Bessel’s equation and suppose that \( \nu \) is a half-integer so that \( \nu \in \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \} \). Find a second solution of Bessel’s equation following Case II(b)(2), p. 72. Be sure to identify the point(s) in your argument where you use the assumption that \( 2\nu \) is odd.

32. Consider Bessel’s equation and suppose that \( \nu \) is an integer. Demonstrate that the smaller root of the indicial equation does not yield a solution.

33. Write Bessel’s equation of order zero in the form

\[ Ly = y'' + \frac{1}{x}y + y = 0. \]

(a) Consider a function in Frobenius form

\[ y(x; \alpha) = \sum_{n=0}^{\infty} a_n(\alpha) x^{n+\alpha} \]

for \( a_0(\alpha) \neq 0 \). Calculate \( Ly \).

(b) Find the indicial equation and the recursion relations for \( a_n(\alpha) \).

(c) Show that the indicial equation has a repeated root; call it \( \alpha_0 \).

(d) Solve the recursion relations without solving the indicial equation. Call the resulting function \( y_0(x; \alpha) \). Assume that \( \alpha > -1 \) in your argument, and note the point(s) when this assumption is used.

(e) Explain why \( L\frac{\partial}{\partial \alpha} y_0(x; \alpha_0) = 0 \).

(f) Calculate \( L\frac{\partial}{\partial \alpha} y_0(x; \alpha) \) for general \( \alpha \); be sure to simplify. Compare your result to (3.3.11).

(g) Conclude that \( \alpha \mapsto L\frac{\partial}{\partial \alpha} y_0(x; \alpha) \) has a double root at \( \alpha_0 \), for all \( x \).

(h) Conclude that

\[ \frac{\partial}{\partial \alpha}\left(L\left(y_0(x; \alpha)\right)\right)_{\alpha=\alpha_0} = L\left(\frac{\partial}{\partial \alpha}\left(y_0(x; \alpha_0)\right)\right) = 0 \]

and hence that a second solution can be found by calculating \( \partial y_0/\partial \alpha \).

(i) Evaluate

\[ \frac{\partial}{\partial \alpha} a_n(x; \alpha)_{\alpha=\alpha_0} \]

where \( a_n(\alpha) \) solves the recursion relations but not the indicial equation.
(j) Show that
\[ \frac{\partial}{\partial \alpha} y(x; \alpha) = \left( \sum_{n=0}^{\infty} a_n(\alpha)x^{n+\alpha} \right) \ln x + \left( \sum_{n=1}^{\infty} \frac{\partial a_n}{\partial \alpha} x^{n+\alpha} \right) \]
where \( a_0 \) is independent of \( \alpha \).

(k) Use (33h) and (33j) to find a second solution of Bessel’s equation of order zero. Compare your result with p. 572, #2, or the DLMF, 10.8.2.

34. Consider Bessel’s equation of order 1
\[ Ly = y'' + \frac{1}{x} y' + \left( 1 - \frac{1}{x^2} \right) y = 0. \]

(a) Consider a function in Frobenius form
\[ y(x; \alpha) = \sum_{n=0}^{\infty} a_n(\alpha)x^{n+\alpha} \]
for \( a_0(\alpha) \neq 0 \). Calculate \( Ly \).

(b) Find the indicial equation and the recursion relations for \( a_n(\alpha) \).

(c) Show that the indicial equation has two roots \( \alpha_1 \) and \( \alpha_2 \) where \( \alpha_1 - \alpha_2 \) is a positive integer.

(d) Solve the recursion relations without solving the indicial equation; call the result \( y_0(x; \alpha) \).

Make the assumption that \( \alpha > -1 \), and note the point(s) where this is used.

(e) Explain why \( L y_0(x; \alpha_1) = 0 \).

(f) Calculate \( L y_0(x; \alpha) \); be sure to simplify.

(g) Calculate \( \frac{\partial}{\partial \alpha} L y_0(x; \alpha) \). Show that
\[ \frac{\partial}{\partial \alpha} L y_0(x; \alpha) \bigg|_{\alpha = \alpha_1} = \frac{2a_0}{x}, \]
and compare this to (3.3.16).

(h) Solve the equation
\[ L \phi = \frac{2a_0}{x} \]
by supposing that \( \phi \) has the Frobenius form
\[ \phi = \sum_{n=0}^{\infty} c_n x^{n-1}. \]

Compare this approach to the approach of (3.3.17).

(i) Explain why your answer to (34h) can be written with just one term.

(j) Evaluate
\[ \frac{\partial a_n}{\partial \alpha} \bigg|_{\alpha = \alpha_1}. \]

(k) Explain why a second solution \( y_2 \) to Bessel’s equation can be found in the form
\[ y_2 = \phi - \frac{\partial y_0}{\partial \alpha} \bigg|_{\alpha = \alpha_1}. \]

Compare this to (3.3.20).

(l) Find a second solution to Bessel’s equation of order 1. Compare your result with p. 572, #2, or the DLMF, 10.8.2.