Reading: Chapter 1, Sections 1.1 - 1.7.

1. (§1.1). Solve $y' = x^2/y^2$.

2. Let $P_0(x) = x^2 + 4$, $P_1(x) = x$, and $P_2(x) = x^2$. Let $L$ be the linear differential operator

$$L = P_0(x) + P_1(x) \frac{d}{dx} + P_2(x) \frac{d^2}{dx^2}.$$ 

Evaluate $L(\sin x)$.

3. Suppose $y'' + y' - xy = x^2$. Let $u = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Write a first order system for $u$ in the form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = f(x, u_1, u_2).$$

4. (§1.2). For each of the following, determine if the theorem (below) applies.

(a) $y' = \cos(xy)$, $y(0) = 1$.

(b) $y' = e^x + x/y$, $y(1) = 3$.

(c) $y' = |xy|$, $y(2) = 3$.

(d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 + y^2 - t^2 \\ x^2 - y^2 + t^2 \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

**Theorem.** Suppose that $F(x, y)$ and $\partial F/\partial y(x, y)$ are continuous in a neighborhood of $x = x_0$ and $y = y_0$. Then there is a unique function $y$ defined in a neighborhood of $x_0$ so that

$$\begin{cases} 
    y' = f(x, y) \\
    y(x_0) = y_0
\end{cases}$$

5. (§1.3). For each set of functions determine if they are independent on the given interval.

(a) The functions $\{x, x^2\}$ on $[-1, 1]$?

(b) The functions $\{\sin x, \cos x\}$ on $[-\pi, \pi]$?

(c) The functions $\{\sin^2 x, \cos^2 x\}$ on $[-\pi, \pi]$?

The importance of independence in our class is due in part to the following result:

**Theorem.** Given the equation

$$Ly = y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0$$

in any neighborhood where $p_0, p_1, \ldots, p_{n-1}$ are continuous we can find $n$ linearly independent functions $\{y_1, y_2, \ldots, y_n\}$ each satisfying $Ly_i = 0$. Any other solution of $Ly = 0$ is a linear combination of $\{y_1, y_2, \ldots, y_n\}$.

6. Consider the following theorem:

**Theorem.** Given the functions $\{y_1, y_2, \ldots, y_n\}$
(a) If \{y_1, y_2, \ldots, y_n\} are dependent, then \(W[y_1, y_2, \ldots, y_n] = 0\).

(b) If \(W[y_1, y_2, \ldots, y_n] \neq 0\) then \{y_1, y_2, \ldots, y_n\} are independent.

Let \(y_1(x) = x^2\) and \(y_2(x) = x^3\). Calculate \(W[y_1, y_2](x)\). Clearly there are choices of \(x\) for which \(W[y_1, y_2](x) = 0\). Does this contradict the theorem? Why or why not?

7. Consider the equation \(y'' + y = 0\).

(a) Find a pair of linearly independent solutions \(\{y_1, y_2\}\).
(b) Calculate \(W[y_1, y_2](x)\).
(c) Verify that (1.3.5) holds.

8. Consider the functions \(\{y_1, y_2\} = \{e^x, x\}\).

(a) Show that these are independent solutions of \((1-x)y'' + xy' - y = 0\).
(b) Calculate \(w[y_1, y_2](x)\).
(c) Show that (1.3.5) holds.

9. (§1.4). Let \(Lg = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0\) for constants \(p_0, p_1, \ldots, p_{n-1}\), and let \(P(r)\) be the corresponding polynomial

\[P(r) = r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0.\]

(a) Show that for any constant \(r\)

\[L(e^{rx}) = P(r)e^{rx}.\]

(b) Suppose that \(r = r_0\) is a zero of order \(m \geq 2\) of \(P(r)\), so

\[P(r) = (r - r_0)^m Q(r)\]

for some polynomial \(Q(r)\). Show that

\[\left| \frac{d}{dr} L(e^{rx}) \right|_{r=r_0} = 0\]

and

\[\frac{d}{dr} L(e^{rx}) = L(xe^{rx}).\]

Conclude that \(L(xe^{rx}) = 0\).

10. Solve \(y'' - 5y' + 6y = 0\).

11. Solve \(y'' + 6y' + 10y = 0\). [We want real solutions!]

12. Solve the Euler equation \(x^2y'' - 4xy' + 4y = 0\) by making the substitution \(y = x^r\) and solving for \(r\).

13. Consider the equation

\[\frac{d}{dx} f(x, y) = g(x).\]

(a) How would you solve (1) for \(y = y(x)\)?
(b) Show that (1) can be written in the form

\[ P(x, y) + Q(x, y)y' = g(x) \]

for an appropriate choice of functions \( P(x, y) \) and \( Q(x, y) \).

(c) Show that

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

(d) Consider the (exact) equation

\[(x + 1)y' + (y + 1) = 0.\]

i. Show that this has the form \( P(x, y) + Q(x, y)y' = g(x) \).

ii. Show that (c) holds.

iii. Convert this to an equation of the form (1).

iv. Solve.

14. Consider the linear equation

\[ Ly = a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \]

and suppose that \( y_1(x) \) is a function that satisfies \( Ly_1 = 0 \). Write \( y(x) = u(x)y_1(x) \) and find an equation of order 1 for \( u(x) \).

15. (§1.5). Solve \( xy' + y = \sin x \) by writing this as a linear equation, finding the integrating factor, and then finding the solution.

16. Solve \( y'' - 5y' + 6y = e^x \) using variation of parameters.

17. Let \( H(x) \) be the Heaviside function (p.16).

(a) Show that, for any smooth function \( y(x) \) with \( y(x) \to 0 \) and \( x \to \pm \infty \)

\[ \int_{-\infty}^{\infty} H(x-a)y'(x) \, dx = -y(a). \]

(b) Show that, if \( h(x) \) is a smooth differentiable function that

\[ \int_{-\infty}^{\infty} h(x-a)y'(x) \, dx = -\int_{-\infty}^{\infty} h'(x-a)y(x) \, dx. \]

(c) Use the properties of the delta function to show that

\[ \int_{-\infty}^{\infty} \delta(x-a)y(x) \, dx = y(a). \]

(d) Conclude that, in some generalized sense that we have not yet made precise,

\[ \frac{d}{dx} H(x-a) = \delta(x-a). \]

18. The discussion of Green’s functions in the text avoids the question of initial and/or boundary conditions.

Consider the initial-value problem

\[
\begin{aligned}
y'' - 5y' + 6y &= e^x \\
y(0) &= y'(0) = 0
\end{aligned}
\]
(a) Find two independent solutions \( \{y_1, y_2\} \) of the homogeneous problem \( y'' - 5y' + 6y = 0 \).

(b) Write out \( G(x; a) \) in the form

\[
G(x; a) = \begin{cases} 
A_1y_1 + A_2y_2 & x < a \\
B_1y_1 + B_2y_2 & x > a 
\end{cases}
\]

then write out (1.5.16) and (1.5.17) for the choice you made in (a).

(c) Impose the conditions

\[
G(x; a) \bigg|_{x=0} = \frac{\partial G}{\partial x} \bigg|_{x=0} = 0
\]

and solve for \( G(x; a) \).

(d) Use this choice of \( G \) and (1.5.13) to solve (2).

(e) To emphasize the significance of the boundary conditions, what is the function \( G \) specified by (1.5.18)? Is it the same as the solution you obtained in (c)?

(f) Does the choice of Green’s function in (1.5.18) satisfy the initial condition in (2)?

(g) What is the value of (1.5.13) for the choice of \( G \) given by (1.5.18)?

19. (§1.6). Solve the Bernoulli equation \( xy' + 2y = x^2 \sqrt{y} \).

20. Solve the equation \( y' = \cos(x + y) \) by making the substitution \( u = x + y \).

21. (§1.7). Write \( yy'' + y' + y^2 = 0 \) as a first order equation.

22. Convert \( x^2y'' = xyy' \) into an autonomous equation.