Math 674
Assignment #8
Due October 24, 2012

Reading: No new reading.

1. 2.3.3.a

2. 2.3.3.b. For simplicity, only consider the case where $n$ is even.

3. 2.3.5

4. Legendre’s differential equation is

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$ 

Use the method of Frobenius to show that the general solution is

$$y = c_0 \left[ 1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 - \cdots \right]
+ c_1 \left[ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 - \cdots \right].$$

Hint: $n(n+1) - k(k+1)$ can be factored.

5. Prove Rodrigue’s Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$ 

Hint: Integrate $P_n(x)$ repeatedly. What is the series representation of $(x^2 - 1)^n$?

6. Prove the generating function identity

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$ 

Hint: Use the binomial theorem on the left.

7. Prove the orthogonality of the Legendre polynomials; in particular prove that

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = 0$$

for $m \neq n$. Hint: Use the same technique that worked to prove the orthogonality of $J_n(\lambda_k x)$.

8. Prove that

$$\int_{-1}^{1} P_n^2(x) \, dx = \frac{2}{2n+1}.$$ 

Hint: Square both sides of Rodrigue’s formula, and integrate, using the previous problem. You will need to know the series for $\ln(1 \pm x)$.

9. Let

$$f(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$

Write $f(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 P_3(x) + A_4 P_4(x) + A_5 P_5(x) + \cdots$. 


10. Solve $\triangle u = 0$ inside the sphere of radius 1 \{$(\rho, \theta, \phi) : \rho \leq 1$\} where $u = u_0$ for $\rho = 1$ and $0 \leq \phi < \pi/2$ while $u = 0$ for $\rho = 1$ and $\pi/2 < \phi \leq \pi$. Assume that the solution is bounded.

Recall that, in spherical coordinates $\{\rho, \theta, \phi\}$ with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$ that

$$\triangle u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2 \sin \phi} \frac{\partial^2 u}{\partial \theta^2}$$

11. Solve $\triangle u = 0$ outside the sphere of radius 1 \{$(\rho, \theta, \phi) : \rho \geq 1$\} where $u = u_0$ for $\rho = 1$ and $0 \leq \phi < \pi/2$ while $u = 0$ for $\rho = 1$ and $\pi/2 < \phi \leq \pi$. Assume that the solution is bounded.