Read: 2.1 and 2.2 (Pages 61-71.)

1. Prove that the operator \( u \mapsto \Delta u \) in \( \mathbb{R}^n \) is rotationally symmetric, that is if \( A \) is an \( n \times n \) orthogonal matrix and if \( y = Ax \), then

\[
\left( \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2} \right) u = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) u
\]

2. Prove that, in \( n \)-dimensions, if \( u \) depends only on \( r = (x_1^2 + \cdots + x_n^2)^{1/2} \) then

\[
\Delta u = n - 1 \cdot u_r + u_{rr}.
\]

3. Solve 1.7.1

4. Prove that

\[
\int_0^1 x J_n(\lambda x)J_n(\mu x) \, dx = \frac{\mu J_n(\lambda)J'_n(\mu) - \lambda J_n(\mu)J'_n(\lambda)}{\lambda^2 - \mu^2}
\]

for \( \lambda \neq \mu \).

HINT: Let \( y_1 = J_n(\lambda x) \) and \( y_2 = J_n(\mu x) \). Find differential equations that \( y_1 \) and \( y_2 \) satisfy. Multiply the equation for \( y_1 \) by \( y_2 \) and vice-versa; then add. Look for an exact derivative.

5. Prove that

\[
\int_0^1 x J^2_n(\lambda x) \, dx = \frac{1}{2} \left\{ [J'_n(\lambda)]^2 + \left( 1 - \frac{n^2}{\lambda^2} \right) J^2_n(\lambda) \right\}
\]

HINT: Pass to the limit \( \mu \to \lambda \) in the previous problem.

6. Let \( A \) and \( B \) be constants, not both zero, and suppose that \( \lambda \) and \( \mu \) are different roots of

\[
AJ_n(x) + BJ'_n(x) = 0.
\]

Prove that

\[
\int_0^1 x J_n(\lambda x)J_n(\mu x) \, dx = 0.
\]

7. Suppose that

\[
f(x) = \sum_{k=1}^{\infty} c_k J_n(\lambda_k x)
\]

for \( 0 < x < 1 \), where \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the positive roots of \( J_0(x) = 0 \).

Prove that

\[
c_k = \frac{2}{J'_{n+1}(\lambda_k)} \int_0^1 x J_n(\lambda_k x)f(x) \, dx.
\]

8. Prove that

\[
\sum_{k=1}^{\infty} \frac{J_0(\lambda_k x)}{\lambda_k \cdot J_1(\lambda_k)} = \frac{1}{2}
\]

for any \( x \), where \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the positive roots of \( J_0(x) = 0 \).
9. Prove that
\[ x^2 = \sum_{k=1}^{\infty} \frac{2(\lambda_k^2 - 4)J_0(\lambda_k x)}{\lambda_k^2 J_1(\lambda_k)} \]
for any \( x \), where \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the positive roots of \( J_n(x) = 0 \).

10. Use separation of variables on the problem
\[
\begin{cases}
  u_t - \Delta u = 0 \\
  u|_{t=0} = u_0 \\
  u|_{r=1} = 0
\end{cases}
\]
on the disc \( \{ (r, \theta) : 0 \leq r \leq 1 \} \). Show that
\[
u(r, t) = \sum_{k=1}^{\infty} \frac{2u_0}{\lambda_k J_1(\lambda_k)} J_0(\lambda_k r) e^{-\lambda_k^2 t} \]
Include all details.

11. Use separation of variables to solve the problem
\[
\begin{cases}
  u_t - \Delta u = 0 \\
  u|_{t=0} = u_0 r^2 \\
  u|_{r=1} = 0
\end{cases}
\]
on the disc \( \{ (r, \theta) : 0 \leq r \leq 1 \} \).