1. Let $Lu = -u''$ with boundary conditions $u(0) = u(1) = 0$. Find the eigenvalues and eigenfunctions for this operator. In particular, $\lambda$ is an eigenvalue and $u$ is a corresponding eigenfunction if $u$ is a nonzero solution of

\[
Lu = \lambda u \\
u(0) = u(1) = 0.
\]

2. Find the Green’s functions $G_\lambda(x, \xi)$ for the operators $L_\lambda u = -u'' - \lambda u$ with boundary conditions $u(0) = u(1) = 0$ where $\lambda$ is not an eigenvalue. [Note that you can restrict your attention only to positive values of $\lambda$.]

3. The Fredholm Alternative is an important result in the study of ordinary and partial differential equations. It is motivated by the following linear algebraic fact. If $A$ is a symmetric $n \times n$ matrix, then either the equation

\[
Ax = b
\]

has a solution $x$ for every choice of $b$, or there exists a vector $y$ with $Ay = 0$, in which case the equation (1) has a solution if and only if $\langle b, y \rangle = 0$ for every $y$ with $Ay = 0$.

How does the Fredholm Alternative explain your answers for problem 2?

4. Find the Green’s function $G(x, \xi)$ for the problem

\[
-[(1 - x^2)u']' = 0, \\
u(0) = 0, \\
\lim_{x \uparrow 1}(1 - x^2)u'(x) = 0.
\]