1. Prove that
\[ \int_0^1 x J_n(\lambda x) J_n(\mu x) \, dx = \frac{\mu J_n(\lambda) J'_n(\mu) - \lambda J_n(\mu) J'_n(\lambda)}{\lambda^2 - \mu^2} \]
for \( \lambda \neq \mu \).

**Hint:** Let \( y_1 = J_n(\lambda x) \) and \( y_2 = J_n(\mu x) \). Find differential equations that \( y_1 \) and \( y_2 \) satisfy. Multiply the equation for \( y_1 \) by \( y_2 \) and vice-versa; then add. Look for an exact derivative.

2. Prove that
\[ \int_0^1 x J_n^2(\lambda x) \, dx = \frac{1}{2} \left\{ [J_n'(\lambda)]^2 + \left( 1 - \frac{n^2}{\lambda^2} \right) J_n^2(\lambda) \right\} . \]

**Hint:** Pass to the limit \( \mu \to \lambda \) in the previous problem.

3. Let \( A \) and \( B \) be constants, not both zero, and suppose that \( \lambda \) and \( \mu \) are different roots of
\[ AJ_n(x) + BJ'_n(x) = 0. \]

Prove that
\[ \int_0^1 x J_n(\lambda x) J_n(\mu x) \, dx = 0 . \]

4. Suppose that
\[ f(x) = \sum_{k=1}^{\infty} c_k J_n(\lambda_k x) \]
for \( 0 < x < 1 \), where \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the positive roots of \( J_n(x) = 0 \).

Prove that
\[ c_k = \frac{2}{J_{n+1}^2(\lambda_k)} \int_0^1 x J_n(\lambda_k x) f(x) \, dx . \]

5. Prove that
\[ \sum_{k=1}^{\infty} \frac{J_0(\lambda_k x)}{\lambda_k J_1(\lambda_k)} = \frac{1}{2} \]
for any \( x \), where \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the positive roots of \( J_n(x) = 0 \).

6. Prove that
\[ x^2 = \sum_{k=1}^{\infty} \frac{2(\lambda_k^2 - 4)J_0(\lambda_k x)}{\lambda_k^2 J_1(\lambda_k)} \]
for any \( x \), where \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the positive roots of \( J_n(x) = 0 \).
7. Use separation of variables on the problem
\[
\begin{align*}
  \frac{\partial u}{\partial t} - \Delta u &= 0 \\
  u|_{t=0} &= u_0 \\
  u|_{r=1} &= 0
\end{align*}
\]
on the disc \( \{(r, \theta) : 0 \leq r \leq 1\} \). Show that
\[
u(r, t) = \sum_{k=1}^{\infty} \frac{2u_0}{\lambda_k J_1(\lambda_k)} J_0(\lambda_k r) e^{-\lambda_k^2 t}.
\]
Include all details.

8. Use separation of variables to solve the problem
\[
\begin{align*}
  \frac{\partial u}{\partial t} - \Delta u &= 0 \\
  u|_{t=0} &= u_0 r^2 \\
  u|_{r=1} &= 0
\end{align*}
\]
on the disc \( \{(r, \theta) : 0 \leq r \leq 1\} \).