Math 674
Assignment #1
Green’s Functions
Order Relations
Due September 12, 2005

Name________________________

1. Let $Lu = -u''$ with boundary conditions $u(0) = u(1) = 0$. Show that $L$ has eigenvalues $\lambda_n = n^2\pi^2$ for $n = 1, 2, \ldots$ with eigenfunctions $u_n(x) = \sin n\pi x$. This means that

$$Lu_n = \lambda_n u_n$$
$$u_n(0) = u_n(1) = 0.$$ 

2. Find the Green’s functions $G_{\lambda}(x, \xi)$ for the operators $L_{\lambda}u = -u'' - \lambda u$ with boundary conditions $u(0) = u(1) = 0$. For simplicity, solve only the case for $\lambda > 0$.

3. The Fredholm Alternative is an important result in the study of ordinary and partial differential equations. It is motivated by the following linear algebraic fact. If $A$ is a symmetric $n \times n$ matrix, then either the equation

$$Ax = b$$

has a solution $x$ for every choice of $b$, or there exists a vector $y$ with $Ay = 0$, in which case the equation (1) has a solution if and only if $\langle b, y \rangle = 0$ for every $y$ with $Ay = 0$.

How does the Fredholm Alternative explain the relationship between your answers for problems 1 and 2?

4. Find the Green’s function $G(x, \xi)$ for the problem

$$-[ (1 - x^2) u' ]' = 0,$$
$$u(0) = 0,$$
$$\lim_{x \to 1} (1 - x^2) u'(x) = 0.$$ 

5. Prove that $e^{-cx} - 1 = O(\epsilon)$ as $\epsilon \downarrow 0$ uniformly for $x \in [0, 1]$.

6. Suppose that $u(x; \epsilon)/v(x; \epsilon)$ is defined for all $x \in \Omega$ and for all $\epsilon > 0$. Prove that $u(x; \epsilon) = o(v(x; \epsilon))$ if and only if

$$\lim_{\epsilon \downarrow 0} \frac{u(x; \epsilon)}{v(x; \epsilon)} = 0$$

for all $x \in \Omega$.

7. Prove that if $u(x; \epsilon) = o(v(x; \epsilon))$ then $u(x; \epsilon) = O(v(x; \epsilon))$.

8. Prove that if $u(x; \epsilon) \sim 0$ then $u(x; \epsilon) = 0$. 