Math 673
Assignment #6
Applications of Fourier Transforms
Due March 9, 2005

Name

1. The Cauchy problem for the Klein-Gordon equation is

\[ u_{tt} - c^2 u_{xx} + a^2 u = 0 \quad \text{for } -\infty < x < \infty, \quad t > 0 \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad \text{for } -\infty < x < \infty. \]

Find a formula that represents the solution. Explain why

\[
\int_{-\infty}^{\infty} |u(x, t)|^2 \, dx = \int_{-\infty}^{\infty} \left( \hat{f}(\xi) \cos(t\sqrt{c^2 \xi^2 + a^2}) + \frac{\hat{g}(\xi)}{\sqrt{c^2 \xi^2 + a^2}} \sin(t\sqrt{c^2 \xi^2 + a^2}) \right)^2 \, d\xi.
\]

Conclude that, for any time \( t \)

\[
\int_{-\infty}^{\infty} |u(x, t)|^2 \, dx \leq 4 \int_{-\infty}^{\infty} |f(x)|^2 \, dx + 4t^2 \int_{-\infty}^{\infty} |g(x)|^2 \, dx.
\]

2. For \( 0 < \alpha < 1 \), show that

\[
F_{c^{\alpha-1}x^{-\alpha}} = \sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\xi^\alpha} \cos \left( \frac{\alpha \pi}{2} \right).
\]

Hint: Replace \( \cos x\xi \) by its sum of exponentials, use Cauchy’s theorem and the contours

\[ iR \quad -iR \]
\[ i\rho \quad -i\rho \]
\[ \rho \quad R \]

as appropriate.

3. Use the previous problem to conclude

\[
F_{c^{\alpha-1}x^{-p}} = \sqrt{\frac{2}{\pi}} \frac{\Gamma(p-1)}{\xi^{p-1}} \sec \left( \frac{p\pi}{2} \right)
\]

for \( 0 < p < 1 \).

4. Find

\[
\mathcal{F}_s \left( \frac{x}{\sqrt{a^2 - x^2}} H(a - x) \right).
\]

Not so much a hint as a potentially useful cryptic remark: A hint from a previous homework assignment may prove useful.
5. Find $\mathcal{F}_c(H(b - x))$. Use it to evaluate
\[ \int_0^\infty \frac{\sin bx}{x(x^2 + a^2)} \, dx. \]

6. Prove that $\mathcal{F}_c\{f'(x)\} = \xi \mathcal{F}_s f - \sqrt{\frac{2}{\pi}} f(0)$ and that $\mathcal{F}_s\{f'(x)\} = -\xi \mathcal{F}_c f$.

7. Find $\mathcal{F}_c\{e^{-\alpha x^2}\}$ for $\alpha > 0$. Use that result and properties of the derivatives of Fourier sine and cosine transforms to find $\mathcal{F}_s\{xe^{-\alpha x^2}\}$. 