1. Show that the solution of
\[ u_t = \kappa u_{xx} \quad 0 \leq x \leq a \]
\[ u\big|_{x=0} = f(t) \]
\[ u\big|_{x=a} = 0 \]
\[ u\big|_{t=0} = 0 \]
is
\[ u(x, t) = \frac{2\kappa \pi}{a^2} \sum_{n=1}^{\infty} n \sin \left( \frac{n\pi x}{a} \right) \int_0^t f(t - \tau) \exp \left\{ -\frac{n^2\pi^2\kappa}{a^2}\tau \right\} d\tau. \]
Hints: Derive and use an identity for \( \sinh(x - y) \). You will also need the identity for \( \sin(x - y) \).

2. Show that the solution of
\[ u_{xx} = \frac{1}{c^2} u_{tt} + k \quad 0 \leq x \leq \ell \]
\[ u\big|_{t=0} = u_t\big|_{t=0} = 0 \]
\[ u\big|_{x=0} = u\big|_{x=\ell} = 0 \]
is the inverse Laplace transform of
\[ \mathcal{U}(x, s) = \frac{c^2 k}{s^3} \left[ \frac{\cosh \left( \frac{s(\ell-x)}{c} \right)}{\cosh \left( \frac{s\ell}{c} \right)} - 1 \right] \]
Conclude that
\[ u(x, t) = \frac{k}{2} x(x - 2\ell) + \frac{16\ell^2 k}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos \left( \frac{2n+1}{2} \pi \frac{\ell-x}{\ell} \right) \cos \left( \frac{c(2n+1)_n\ell}{\ell} \right) \]
Hint: Despite appearances, \( \mathcal{U} \) has a pole of order 1 at the origin, not a pole of order 3.

3. Solve
\[ y(t) + \int_0^t e^{-\tau} y(t - \tau) d\tau = 1. \]

4. Apply the method of Frobenius to Bessel’s equation of order \( \nu \)
\[ x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \]
Show that a multiple of the first solution so obtained is \( J_\nu(x) \).

5. If \( \nu \) is not an integer, show that a second independent solution of Bessel’s equation of order \( \nu \) is \( J_{-\nu}(x) \).
6. If \( \nu \) is an integer, show that a second independent solution of Bessel’s equation of order \( \nu \) is

\[
Y_{\nu}(x) = \frac{2}{\pi} \left\{ \ln \left( \frac{x}{2} \right) + \gamma \right\} J_{\nu}(x) - \frac{1}{\pi} \sum_{n=0}^{\nu-1} \frac{(\nu-n-1)!}{n!} \left( \frac{x}{2} \right)^{2n-\nu} \]

\[
- \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\nu)} \{H_n + H_{n+\nu}\} \left( \frac{x}{2} \right)^{2n-\nu}.
\]

Here \( \gamma = \lim_{m \to \infty} (H_m - \ln m) \) is Euler’s constant.

Hints on (4)–(6):

- \( a_{2n} = \frac{(-1)^n a_0}{[(r+2+\nu)(r+4+\nu) \ldots (r+2n+\nu)][(r+2-\nu)(r+4-\nu) \ldots (r+2n-\nu)]} \)

- \( c_{2n} = \frac{\partial}{\partial r} \left\{ \frac{(-1)^n (r+\nu)a_0}{[(r+2+\nu)(r+4+\nu) \ldots (r+2n+\nu)][(r+2-\nu)(r+4-\nu) \ldots (r+2n-\nu)]} \right\}_{r=-\nu} \)

- The form of \( c_{2n} \) depends on whether or not \( n > \nu \). In the first case, the expression for \( c_{2n} \) will have some cancellation while in the latter case it will not.

- If \( y_1(x) \) and \( y_2(x) \) are independent solutions of a differential equation, then so are \( y_1(x) \) and \( ay_1(x) + by_2(x) \) for any numbers \( a \) and \( b \) with \( b \neq 0 \).