8. A ring is a structure consisting of a non-empty set $R$, two binary operations $\oplus$ and $\otimes$, and two elements named $0, 1 \in R$ so that

(a) Properties of $\oplus$
- For any $x, y \in R$, we have $x \oplus y \in R$.
- For any $x, y, z \in R$, we have $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
- For any $x, y \in R$, we have $x \oplus y = y \oplus x$.
- For any $x \in R$, we have $x \oplus 0 = 0 \oplus x = x$.
- For any $x \in R$, there is a $y \in R$ so that $x \oplus y = 0$.

(b) Properties of $\otimes$
- For any $x, y \in R$, we have $x \otimes y \in R$.
- For any $x, y, z \in R$, we have $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.
- For any $x \in R$, we have $1 \otimes x = x \otimes 1 = x$.

(c) Distributive properties
- For any $x, y, z \in R$, we have $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.
- For any $x, y, z \in R$, we have $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$.

It is usual to replace the special symbols $\oplus$ and $\otimes$ with the more common $+$ and $\cdot$ whenever it would not cause confusion.

Do the Gaussian integers $\mathbb{Z}[i]$ with the operations described in the text form a ring?

9. The text claims that the Gaussian integer $-3 + 2i$ can be found by rotating the Gaussian integer $2 + 3i$ through $90^\circ$ to the right. Prove this.

10. For a Gaussian integer $a + bi$, define its modulus by $|a + bi| = \sqrt{a^2 + b^2}$. What is the geometric interpretation of the modulus? If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ are Gaussian integers, show that $|z_1 z_2| = |z_1| |z_2|$. 

11. Is the collection of circular numbers $\mathbb{Z}_n$ described in the text a ring?

12. Given a ring $R$, an element $x \in R$ is called a zero divisor if the ring contains an element $y$ so that $xy = 0$. Does $\mathbb{Z}_3$ contain a zero divisor? $\mathbb{Z}_4$? $\mathbb{Z}_5$? $\mathbb{Z}_6$? $\mathbb{Z}_7$? $\mathbb{Z}_8$?

13. Given a ring $R$, an element $x \in R$ is called a (left) unit if every element of the ring $R$ is a (left) multiple of $x$. In particular, this means that, for every $r \in R$, there is some $y \in R$ so that $yx = r$.

In $\mathbb{Z}_5$, find all of the multiples of 3. Do the same in $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, and $\mathbb{Z}_9$.

14. Imitate the proof of Theorem 1 and show that $\sqrt{3}$ is irrational.

15. Find six units in $\mathbb{Z}[\sqrt{2}]$. 