1. What is a number? What are the essential properties it should possess?

2. Let \( f(x) = 1/x \) and \( g(x) = 1 - x \).
   
   (a) How many functions can be obtained by composing \( f \) and \( g \) arbitrarily often?
   
   (b) List all such functions.
   
   (c) Construct a “multiplication table” for all of the compositions.
   
   (d) Is your “multiplication” associative? Is it commutative?

3. Consider an equilateral triangle \( T \) in the plane, and consider the set of all isometries of the plane that preserve \( T \). [An isometry is a function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) so that for any pair of points \( x, y \in \mathbb{R}^2 \), we have \( |x - y| = |f(x) - f(y)| \); i.e. the function \( f \) preserves the distance between points.]
   
   (a) Show that if \( f \) is an isometry of the plane that preserves \( T \) then vertices of the triangle are mapped to vertices. [Hint: What is the distance between the vertices?]
   
   (b) Identify all such isometries.
   
   (c) Construct a “multiplication table” for all of the compositions.
   
   (d) Is your “multiplication” associative? Is it commutative?

4. Consider the collection of all permutations of the numbers \( \{1, 2, 3\} \). A permutation is a function \( f : \{1, 2, 3\} \to \{1, 2, 3\} \) with the property that is both one-to-one and onto.
   
   (a) Identify all such permutations.
   
   (b) Construct a “multiplication table” where permutations are combined by composition.
   
   (c) Is your “multiplication” associative? Is it commutative?
   
   (d) What is the relationship between this multiplication table and the one found in the previous problem?

5. Consider the set \( \{0, 1, 2, 3, 4, 5\} \), and let \( \oplus \) be addition mod 6, so that if \( x + y = z \) and \( x \oplus y = \zeta \), then \( \zeta \in \{0, 1, 2, 3, 4, 5\} \) and \( z - \zeta \) is a multiple of 6.
   
   (a) Construct an “addition table” for \( \oplus \).
   
   (b) Is your “addition” associative? Is it commutative?

6. Consider the set \( \{1, 2, 3, 4, 5, 6\} \), and let \( \otimes \) be multiplication mod 7, so that if \( xy = z \) and \( x \otimes y = \zeta \), then \( \zeta \in \{1, 2, 3, 4, 5, 6\} \) and \( z - \zeta \) is a multiple of 7.
   
   (a) Construct a “multiplication table” for \( \otimes \).
   
   (b) Is your “multiplication” associative? Is it commutative?

7. These examples have shown that the notion of “number” can be extended to a range of scenarios, from functions, to permutations, to geometric transformations. Can you identify other potential examples?