Comprehension

1) Matrix Methods
   a) What is the relationship between the $LU$ factorization of a matrix and Gaussian elimination?
   b) What is pivoting in Gaussian elimination, and why is it important?
   c) You want to solve the following system using iterative methods:

\[
\begin{pmatrix}
4 & 1 & 0 & 1 \\
0 & 4 & 1 & 1 \\
-1 & 1 & 4 & 0 \\
0 & 2 & -1 & 4
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix}
1 \\
-5 \\
11 \\
12
\end{pmatrix}.
\]

Which method do you select, and why? What happens if the “2” is replaced by “12”?

2) Numerical Approximation
   a) What is the Newton form of the interpolating polynomial? What is the Lagrange form? Which is more useful in practice?
   b) Are high order interpolating polynomials used in practice? Why or why not?
   c) What is a spline?

3) Differential Equations
   a) What are the relative advantages and disadvantages of Taylor methods, Runge-Kutta methods, Adams-Bashforth methods, Adams-Moulton methods, and predictor corrector methods?
   b) How would you solve the second order equation

\[
\begin{align*}
\frac{d^2y}{dt^2} + 2t \frac{dy}{dt} + (2 - t^2)y &= 0, \\
y(0) &= 1, \\
y'(0) &= 2
\end{align*}
\]

What would you do differently in the previous problem if we replace the condition $y'(0) = 2$ with the condition $y(2) = 2$?

4) Partial Differential Equations
   a) What are the three main types of partial differential equations? Give an example of each.
   b) When solving equations of the form $u_t = u_{xx}$, by finite difference methods, can we choose the spatial grid spacing and the time grid spacing independently of one another? Explain.

5) Eigenvalues
   a) What is the direct method for calculating the eigenvalues and eigenvectors of a matrix? Illustrate with an example.
   b) Is the direct method for calculating the eigenvalues and eigenvectors of a matrix practical? Why or why not?
   c) What is the power method for calculating the eigenvalues and eigenvectors of a matrix?
Computation

6) Suppose that $A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix}$, and $B = \begin{pmatrix} 8 \\ 7 \\ 14 \\ -7 \end{pmatrix}$. Give an algorithm that computes the $LU$ factorization of $A$. Explain which type of $LU$ factorization is used. Compute the $LU$ factorization of $A$.

7) Give a graph of the following data and the tension splines through these points for $\tau = 0.1$, $\tau = 1$, and $\tau = 10$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>6</th>
<th>10</th>
<th>13</th>
<th>17</th>
<th>20</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6.67</td>
<td>17.33</td>
<td>42.67</td>
<td>37.33</td>
<td>30.10</td>
<td>29.31</td>
<td>28.74</td>
</tr>
</tbody>
</table>

8) Consider problem \[ \begin{align*}
   y' &= 2 + y^2 - t^3 \\
   y(0) &= 0
\end{align*} \]

   a) Solve this problem using your choice of fourth order method on the interval $[0,2]$.
   b) Name the method that was chosen.
   c) What is the theoretical explanation for the unusual behavior of this solution?

9) Consider the problem \[ \begin{align*}
   u_{xx} &= u_t, \\
   u(x,0) &= \sin \pi x \\
   u(0,t) &= u(1,t) = 0.
\end{align*} \]

   a) Solve this problem on the interval $0 \leq t \leq 1$, with a spatial grid size of $h = 0.1$ using the method of your choice.
   b) Name the method that was chosen.

10) Partial Differential Equations

   Consider the problem \[ \begin{align*}
   u_{xx} + u_{yy} &= 0, \\
   u(x,0) &= 0, \quad u(x,1) = x \quad 0 \leq x \leq 1, \\
   u(0,y) &= 0, \quad u(1,y) = y \quad 0 \leq y \leq 1.
\end{align*} \]

   a) Solve this problem for $h = k = 0.1$ using the method of your choice.
   b) Name the method that was chosen.
   c) How accurate do you expect your solution is?