Choose five (5) problems. If you attempt more than 5, indicate which you wish graded. If you do not make such an indication, the first 5 shall be graded.

1) Interpolation
   a) What is the Lagrange form of an interpolating polynomial? What is the Newton form?
   b) Given a continuous function \( f \), can we always find a sequence of nodes so that the interpolating polynomials converge to \( f \)? If not, explain why not. If so, decide if this fact is of practical value.
   c) Are high order polynomial interpolations used often in practice? Why or why not?
   d) Find an interpolating polynomial through the points \((5,1), (-7,-23), (-6,54), (0,-954)\).

2) Splines.
   a) What is a spline function of degree \( k \)?
   b) Determine whether or not the following function is a natural cubic spline.
      \[
      f(x) = \begin{cases} 
      x^3 - 1 & x \in [-1, \frac{1}{2}], \\
      3x^3 - 1 & x \in [\frac{1}{2}, 1]. 
      \end{cases}
      \]
   c) Give a graph of the following data and the natural cubic spline through these points.
      |   0   |   6   |  10  |  13  |  17  |  20  |  28  |
      |  6.67 | 17.33 | 42.67| 37.33| 30.10| 29.31| 28.74|

   a) State a theorem that guarantees the existence of solutions to the ordinary differential equation
      \[
      \begin{cases} 
      y' = f(t, y) \\
      y(t_0) = y_0 
      \end{cases}
      .
      \]
   b) Show that the problem \( \begin{cases} 
      y' = \tan y \\
      y(0) = 0 
      \end{cases} \) has a solution for \( |t| < \frac{\pi}{4} \).
   c) State a theorem that describes when solutions of the ordinary differential equation \( \begin{cases} 
      y' = f(t, y) \\
      y(t_0) = y_0 
      \end{cases} \)
      are unique.
   d) Find two solutions of \( \begin{cases} 
      y' = y^{1/3} \\
      y(0) = 0 
      \end{cases} \). Explain why this does not contradict your uniqueness result.

4) Runge-Kutta Methods.
   Consider the problem \( \begin{cases} 
      y' = y - t^2 + 1, \\
      y(0) = \frac{4}{3} 
      \end{cases} \)
   a) Find an exact solution of this problem.
   b) Use a fourth order Runge-Kutta method to solve this problem for \( 0 \leq t \leq 2 \). Choose \( h \) sufficiently small that the error is less than 0.002
5) Multistep Methods.
Consider the problem\[
\begin{align*}
y' &= -2xy^2, \\
y(0) &= 1.
\end{align*}
\]
a) Find an exact solution of this problem.
b) Use a fourth order predictor corrector method to solve this problem for \(0 \leq t \leq 1\). Choose \(h\) sufficiently small that the error is less than 0.002.

6) Finite Difference Methods.
Consider the problem\[
\begin{align*}
y'' &= 2y' - y, \\
y(0) &= 1, \\
y(1) &= 2.
\end{align*}
\]
a) Find an exact solution of this problem.
b) Use a second order finite difference method to solve this problem for \(0 \leq t \leq 1\). Choose \(h\) sufficiently small that the error is less than 0.01.