Take Home Final Exam
Math 635
Due: December 18, 2002 at 7:00.
*The due date will not be extended*

Name_________________________

1. Solve the initial-boundary value problem

\[
u_t = u_{xx} \quad 0 \leq x \leq \pi, t > 0
\]
\[
u(x,0) = 0
\]
\[
u(0,t) = 0
\]
\[
u(\pi,t) = 1
\]

Use any method you wish. Give a good graph of the solution at time \( t = 1 \). The exact solution is

\[
u(x,t) = \frac{1}{\pi} \left[ x + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin nx \right].
\]
Ensure that your error is no more than 0.01. Explain how you did so.

2. Solve the initial-boundary value problem

\[
\frac{\partial u}{\partial t} = 0.15 \left[ \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right] \quad 0 \leq r < 1, t > 0
\]
\[
u(1,t) = \sin 10t
\]
\[
u(r,0) = \sin r^2
\]

Use any method you wish. Give a good graph of the solution at time \( t = 0.5 \).

3. The DuFort-Frankel scheme for solving the heat equation \( u_t = u_{xx} \) is

\[
\frac{u_{m+1}^n - u_{m-1}^{n-1}}{2k} = \frac{u_{m+1}^n - (u_{m+1}^{n+1} + u_{m-1}^{n+1})}{h^2} + u_{m-1}^n.
\]

Prove the following:

a. This method is accurate of order \( O(h^2) + O(k^2) + O\left(\frac{k^2}{h^2}\right) \).

b. This method is explicit. [Show that you can write \( u_{m+1}^{n+1} \) solely in terms of \( u_j^n \) for \( j \leq n \).]

c. This finite difference scheme is a *multi-step* scheme, because it uses data \( u^n \) and \( u^{n-1} \) to calculate the values of \( u^{n+1} \). When one performs the Von Neumann stability analysis on a multi-step scheme, the equation for the amplification polynomial \( g \) is quadratic, and you obtain two solutions \( g_{\pm} \). The method is *stable* if

i. \( |g_{\pm}(\theta)| \leq 1 \) and

ii. if \( |g_{\pm}(\theta)| = 1 \), then \( g(\theta) \) is a simple root.

Show that this method is unconditionally stable.

d. Show that if \( \lambda = k/h \) is held constant, then the scheme is consistent with \( \lambda^2 u_{xx} + u_t = u_{xx} \).

e. What is the practical consequence of d.?
4. Solve the initial-boundary value problem

\[ u_t + u_x = 0 \quad 0 \leq x \leq 2, \quad t > 0 \]

\[ u(x,0) = \begin{cases} x(1-x) & 0 \leq x \leq 1 \\ 1-x & 1 \leq x \leq 2 \end{cases} \]

\[ u(0,t) = t \]

Use any method. Give good graphs of the solution at times \( t = 0.25 \), \( t = 0.5 \), \( t = 0.75 \), and \( t = 1 \). Describe the behavior of the solution.