Take Home Exam #1
Math 635
Due: October 16, 2002

Name_________________________

1. Solve the initial value problem
\[
\begin{align*}
y' &= y - t^2 + 1 \\
y(0) &= 1
\end{align*}
\]
on the interval \(0 \leq t \leq 5\), using Euler’s method, Implicit Euler with iteration, and implicit Euler with Newton’s method. Explain your results.

2. Solve the system
\[
\begin{align*}
y_1' &= 9y_1 + 24y_2 + 5\cos t - \frac{1}{3}\sin t \\
y_2' &= -24y_1 - 51y_2 - 9\cos t + \frac{1}{3}\sin t \\
y_1(0) &= \frac{4}{3} \\
y_2(0) &= \frac{2}{3}
\end{align*}
\]
on the interval \(0 \leq t \leq 10\), using Euler’s method, Implicit Euler with iteration, and implicit Euler with Newton’s method. Explain your results.

3. Consider the system
\[
\begin{align*}
y_1' &= \alpha - y_1 - \frac{4y_1y_2}{1 + y_1^2} \\
y_2' &= \beta y_1 \left( 1 - \frac{y_2}{1 + y_1^2} \right) \\
y_1(0) &= 0 \\
y_2(0) &= 2
\end{align*}
\]
Fix \(\alpha = 10\). Choose any method, and plot the solution for \(0 \leq t \leq 20\) for \(\beta = 2\) and then for \(\beta = 4\). In each case plot \(y_1\) vs. \(t\), \(y_2\) vs \(t\), and \(y_1\) vs \(y_2\). Describe your results.

Investigate the situation near the value \(\beta = 3.5\). You may need to extend the interval of integration.

4. Consider the method
\[
y_n = y_{n-1} + hf \left( t_{n-1/2}, \frac{1}{2} (y_{n-1} + y_n) \right)
\]
called the implicit midpoint method.
   a. Show that the implicit midpoint method is symmetric, second-order, and A-stable.
   b. Show that even if we allow the eigenvalue \(\lambda\) to vary in \(t\) so that \(y' = \lambda(t)y\) in the test equation, we still have \(|y_n| \leq |y_{n-1}|\) if Re \(\lambda \leq 0\). [This property is called \(AN\)-stability.] Show that the trapezoidal rules is not \(AN\)-stable.