Final Examination
Math 581
December 20, 1999

Name_________________________

You have 2 hours. All problems are worth the same amount. The use of graphing calculators is permitted.

§1 Computation

1) Calculate the following limits
   a) \( \lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 2x} \)
   b) \( \lim_{x \to 0} \frac{e^{2x} - 1}{x} \)
   c) \( \lim_{x \to 0} (\cos x)^{1/x^2} \)

2) Calculate the derivative of each of the following functions.
   a) \( y(x) = \frac{\cos x}{x^2 + 1} \)
   b) \( y(x) = \ln \left( 1 - \frac{1}{2 + \cos x} \right) \)
   c) \( y(x) = \tan^{-1} \left( \frac{1}{1 + x^2} \right) \)

3) Calculate the following.
   a) \( \int_0^1 x \ln(x + 3) \, dx \)
   b) \( \int \frac{dx}{x^2 \sqrt{x^2 - 1}} \).
   c) \( \int \ln x \, dx \).

4) What is the Divergence Theorem? Let \( Q \) be the solid region between the paraboloid \( z = 4 - x^2 - y^2 \) and the xy-plane. Verify the Divergence Theorem for the vector field \( \vec{F} = \left( 2z, x, y^2 \right) \).

5) The Bessel function of order 1 is given by \( J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} \). What is the radius of convergence of this series? Show that \( J_1(x) \) solves the equation \( x^2 y'' + xy' + (x^2 - 1)y = 0 \).

6) Let \( A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \). Show that the null space \( \{X : AX = 0\} \) is a vector space, and find a basis for the null space. Show that the range \( \{B : AX = B\} \) is a vector space, and find a basis for the range.

7) Let \( A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \). Find all of the eigenvalues of \( A \) and the corresponding eigenspaces.
8) Solve:
   a) \( yy' - e^x = 0 \), \( y(0) = 1 \).
   b) \( y' + y = \sin x \), \( y(\pi) = 1 \).
   c) \( y'' - y' - 2y = e^{3x} \).

§2 Comprehension

9) Give a precise definition of the limit \( \lim_{x \to a} f(x) = L \). Use the precise definition to prove that
   \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0 \).

10) Give a precise definition of the derivative of a function. Let \( f(x) = \sqrt{2x-1} \). Evaluate \( f'(5) \) from the definition.

11) Prove that \((1 + x)^n > 1 + nx \) for \( n = 2, 3, \ldots \) if \( x > -1 \) and \( x \neq 0 \).

12) What is a vector space? What is a subspace of a vector space? Give two examples of vector spaces whose elements are not members of \( \mathbb{R}^n \) or \( \mathbb{C}^n \). For each example, give a subspace of that vector space that contains at least two elements.

§3 Application

13) In Einstein’s special theory of relativity, the mass of an object moving with velocity \( v \) is
   \[
   m = \frac{m_0}{\sqrt{1 - v^2/c^2}}
   \]
   where \( m_0 \) is the mass of the object at rest, and \( c \) is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest,
   \[
   K = mc^2 - m_0c^2.
   \]
   Show that when \( v \) is small when compared with \( c \), that this expression agrees with the expression from classical Newtonian physics \( K = \frac{1}{2}mv^2 \). Estimate the error in this approximation.

14) According to the Biot-Savart law, and electrical current \( I \) flowing upward in a wire along the \( z \)-axis produces an magnetic field \( \mathbf{H} \) at the point \((x, y, z)\) of the form
   \[
   \mathbf{H}(x, y, z) = \frac{2I}{|\mathbf{r}|}(\mathbf{k} \times \mathbf{r})
   \]
   where \( \mathbf{k} = \langle 0, 0, 1 \rangle \) and \( \mathbf{r} = \langle x, y, 0 \rangle \). Show that if \( C \) is any smooth simple closed curve enclosing the \( z \)-axis, then \( \oint_C \mathbf{H} \cdot d\mathbf{r} = \pm 4\pi I \), while if \( C \) does not enclose the \( z \)-axis, then \( \oint_C \mathbf{H} \cdot d\mathbf{r} = 0 \).

15) What is the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid
   \[
   \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1?
   \]