1) Let \( Q(x, y, z) = 5x^2 + 6y^2 + 7z^2 + 4xy + 4yz \).
   a) Find the symmetric matrix \( M \) which represents \( Q \).
   b) Find the characteristic polynomial of \( M \).
   c) Find the eigenvalues of \( M \).
   d) Find a maximal set of nonzero orthogonal eigenvectors for \( M \).
   e) Find an orthogonal change of coordinates which diagonalizes \( M \).
   f) Identify the quadric surface \( Q(x, y, z) = 1 \).
   g) Graph the surface \( Q(x, y, z) = 1 \).
2) Let $P$ be the inner product space consisting of all polynomials where the inner product is
\[
\langle p(x), q(x) \rangle = \int_{0}^{\infty} e^{-x} p(x) q(x) \, dx.
\]
a) Show that $P$ is an inner product space.

b) The Laguerre polynomials \( \{L_n(x)\}_{n=0}^{\infty} \) are defined by the relationship \( L_n(x) = e^{x} \frac{d^n}{dx^n} \left( x^n e^{-x} \right) \). Find the Laguerre polynomials \( L_0(x), L_1(x), L_2(x), \) and \( L_3(x) \).

c) Show that the subset of the Laguerre polynomials \( \{L_0(x), L_1(x), L_2(x), L_3(x)\} \) form an orthogonal set in $P$.

d) Describe the subspace of $P$ given by \( V = \operatorname{span}\{L_0(x), L_1(x), L_2(x), L_3(x)\} \) in terms of the basis of $P$ given by \( \{1, x, x^2, x^3, \ldots\} \).

e) Find the projection of the function \( f(x) = x^5 \) onto $V$. 

3) Let $A$ be an $n \times n$ real valued matrix.
   a) Show that the set $\{X : AX = 0\}$ is a subspace of $\mathbb{R}^n$. This space is called the \textit{null-space} of the matrix $A$.
   b) Show that the set $\{B : AX = B\}$ is a subspace of $\mathbb{R}^n$. This space is called the \textit{range} of the matrix $A$.
   c) Let $A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 1 & 4 \end{pmatrix}$. Find an orthonormal basis for the null space of $A$.
   d) For the same $A$ as in (c), find an orthonormal basis for the range of $A$. 
4) An $n \times n$ real symmetric matrix $A$ is positive definite if $X^TAX > 0$ whenever $X \neq 0$.
   
a) Show that a matrix is positive definite if $\langle AX, X \rangle > 0$ for all $X \neq 0$, where $\langle u, v \rangle$ is the usual inner product on $\mathbb{R}^n$.
   
b) Prove that the diagonal elements of a positive definite matrix must be positive.
   
c) Prove that the eigenvalues of a positive definite matrix are all positive.
   
d) Show that the determinant of $A$ is positive.
   
e) Show that $A$ is invertible.