Boundary Layers

Suppose that a hot stream of fluid of velocity $U_i$ and temperature $T_0$ flows past a hot plate at $y = 0, x > 0$. The plate is held at the constant temperature $T_1$.

Show that the temperature $T$ satisfies the equation

$$\rho c \frac{\partial T}{\partial x} = k \nabla^2 T$$

with the boundary conditions $T = T_0$ at infinity and $T = T_1$ on $y = 0, x > 0$.

Nondimensionalize the variables, by writing $T = T_0 + (T_1 - T_0)T', x = Lx'$ and $y = Ly'$ where $L$ is an arbitrary length scale. Explain why the variables $T', x'$ and $y'$ are considered dimensionless. Show that the dimensionless variables satisfy $T = 1$ on the plate, $T = 0$ at infinity away from the plate, and

$$\frac{\partial T}{\partial x} = \epsilon \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (1)$$

where for simplicity we have dropped the primes. The number $Pe = \rho c L U / k$ is called the Peclet Number of the problem, and $\epsilon = 1 / Pe$. What is the advantage of writing the problem in this new form?

Define the variables $\xi, \eta$ by the relationship $(\xi + i \eta)^2 = \frac{1}{\epsilon}(x + iy)$ Look for a solution of (1) of the form $T = f(\eta)$. Show that $f$ satisfies

$$f'' + 2\eta f' = 0$$

with $f(\infty) = 0$ and $f(0) = 1$. Conclude that

$$f(\eta) = \text{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-z^2} \, dz$$

Plot a graph or otherwise notice that $\text{erfc} \eta \approx 0$ for $\eta$ large. Thus the region of interest for the solution occurs when $\eta \leq O(1)$. If we examine the solution near the leading edge of the hot plate then $x \leq O(1)$ as well. Use the relationship $(\xi + i \eta)^2 = \frac{1}{\epsilon}(x + iy)$ to conclude that $\xi = O(\epsilon^{-1/2})$. Use this simplification to see that

$$\eta \approx \frac{y}{2\sqrt{\epsilon x}}$$

and hence

$$T \approx \text{erfc} \left( \frac{y}{2\sqrt{\epsilon x}} \right)$$

This shows us that there is a boundary layer of size $y = O(\sqrt{\epsilon x})$ in which $T$ changes from 1 on the plate to 0 in the flow.
Electrodynamics

Let $\mathbf{E}$ be the electric field at a point $x = (x_1, x_2, x_3)$ in space. If the electric field does not change in time (and hence is static), Maxwell’s equations imply that
\[
\nabla \cdot \mathbf{E} = 4\pi \rho,
\]
\[
\nabla \times \mathbf{E} = 0,
\]
where $\rho$ is the electrical charge density in space.

Show that the electric field is conservative. Determine that there is a function $\Phi$, called the electric potential, so that $\mathbf{E} = -\nabla \Phi$. Explain why
\[
\nabla^2 \Phi = -4\pi \rho.
\]

We want to find the electric potential in the region $0 \leq x \leq a$, $y \geq 0$, where $\Phi = 0$ on the sides $x = 0$ and $x = a$, but $\Phi = V$ on $y = 0$. The region is devoid of charge, so $\rho = 0$ throughout. For physical reasons, we also assume that $\Phi \to 0$ for large $y$.

Use separation of variables to show that
\[
\Phi = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-n\pi y/a} \sin \left( \frac{n\pi x}{a} \right)
\]

Let $Z = \exp \left( \frac{2\pi i}{a} (x + iy) \right)$ and explain why
\[
\Phi = \frac{4V}{\pi} \text{Im} \sum_{n \text{ odd}} \frac{Z^n}{n}
\]

Use the Taylor series for $\ln(1 + x) = \sum x^n/n$ to show that
\[
\Phi = \frac{2V}{\pi} \text{Im} \left[ \ln \left( \frac{1 + Z}{1 - Z} \right) \right].
\]

Next, note that the logarithm of the complex number $re^{i\theta}$ is just $\ln r + i\theta$, where $\theta$ is suitably restricted. Thus, we can find the imaginary part of the logarithm of a complex number by finding its argument.

Show that
\[
\frac{1 + Z}{1 - Z} = \frac{1 - |Z|^2}{1 - |Z|^2} + i \frac{2 \text{Im} Z}{1 - |Z|^2}.
\]

Use this to conclude that
\[
\Phi = \frac{2V}{\pi} \tan^{-1} \left( \frac{2 \text{Im} Z}{1 - |Z|^2} \right) = \frac{2V}{\pi} \tan^{-1} \left( \frac{\sin \frac{\pi x}{a}}{\sinh \frac{\pi y}{a}} \right).
\]
The Fundamental Solution of the Heat Equation

The function

$$\Gamma(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

is called the fundamental solution of the heat equation. In this project we shall examine some of the reasons for this name. Prove the following:

- $$\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \Gamma(x, t) = 0.$$  
- $$\int_{-\infty}^{\infty} \Gamma(x, t) \, dx = 1 \text{ for every } t > 0.$$  

Now we will use the fundamental solution of the heat equation to solve the Cauchy Problem

$$\begin{cases} 
  u_t - u_{xx} = 0, & -\infty < x < \infty, \ t > 0 \\
  u(x, t)|_{t=0} = u_o(x) & -\infty < x < \infty.
\end{cases} \tag{2}$$

Let $u_o(x)$ be a bounded continuous function on the whole real line, and define

$$u(x, t) = \Gamma * u_o = \int_{-\infty}^{\infty} \Gamma(x - y, t) u_o(y) \, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} u_o(y) \, dy$$

Use Leibinz’s Rule to prove that $u_t - u_{xx} = 0$.

Now we wish to prove that $\lim_{t\to0} u(x, t) = u_o(x)$. First note that

$$u(x, t) - u_o(x) = \int_{-\infty}^{\infty} \Gamma(x - y, t) [u_o(y) - u_o(x)] \, dy.$$  

Split the integral up into two parts— for $x$ near $y$, and for $x$ far away from $y$. Use the continuity of $u_o$ to estimate the first integral, and use the dominated convergence theorem (p. 687) to estimate the second.

Finally, prove that the solution $u = \Gamma * u_o$ is infinitely differentiable.

It is interesting to note that the solution we have just created is unique in the class of bounded solutions to the Cauchy problem; meaning that there is no other solution of (2) that is also bounded. In fact, there is no other solution $u(x, t)$ of (2) that satisfies an estimate of the form $|u(x, t)| \leq C e^{\alpha x^2}$ for some $C$ and $\alpha$. 
The Calculus of Variations

The length of the curve \( y = f(x) \) between \((0, 0)\) and \((1, 0)\) is

\[
L[f] = \int_0^1 \sqrt{1 + (f'(x))^2} \, dx.
\]

We would like to show that the curve of shortest length connecting these points is a straight line. In particular, we want to find the function \( f \) that minimizes \( L[f] \).

To proceed, let \( \phi \) be any function that satisfies \( \phi(0) = \phi(1) = 0 \). Then for any \( \lambda \), the function \( x \mapsto f(x) + \lambda \phi(x) \) also connects \((0, 0)\) to \((1, 0)\). Since \( f \) is assumed to be the function that generates the minimum length, we know that the function

\[
\lambda \mapsto L[f + \lambda \phi]
\]

has a minimum when \( \lambda = 0 \). Use Leibniz's Rule to evaluate

\[
\frac{d}{d\lambda} L[f + \lambda \phi] \bigg|_{\lambda=0} = \frac{d}{d\lambda} \int_0^1 \sqrt{1 + (f'(x) + \lambda \phi'(x))^2} \, dx \bigg|_{\lambda=0}.
\]

Use integration by parts to conclude that

\[
\int_0^1 \frac{d}{dx} \left( \frac{f'}{\sqrt{1 + (f')^2}} \right) \phi \, dx = 0
\]

for any \( \phi \) with \( \phi(0) = \phi(1) = 0 \). From this, deduce that \( f'(x) = 0 \), and that the shortest curve connecting \((0, 0)\) and \((1, 0)\) is a straight line.

Let \( D \) be any region in the \( xy \)-plane, and let \( f(x, y) \) be any function defined on the boundary \( \partial D \) of \( D \). We want to find the function \( z = u(x, y) \) defined on \( D \) of minimal surface area with \( u|_{\partial D} = f \).

The area of the surface \( z = u(x, y) \) defined over \( D \) is then

\[
A[u] = \iint_D \sqrt{1 + u_x^2 + u_y^2} \, d\sigma.
\]

Use the same techniques as above to show that if \( u \) minimizes this functional over all functions with \( u|_{\partial D} = f \), then

\[
(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0.
\]

This, of course, is the minimal surface equation of Chapter 1.
The Minimal Surface Equation

In this exercise, we show that a minimal surface can not form between two coaxial rings of radius $R$ if they are separated by a distance of more than $1.3255R$.

Consider the minimal surface equation

$$(1 + u_x^2)u_{xx} + (1 + u_y^2)u_{yy} - 2u_xu_yu_{xy} = 0.$$ 

Show that, if $u$ is a radial function, so that $u = f(r)$, then this equation reduces to

$$rf''(r) + f'(r)(1 + [f'(r)]^2) = 0.$$ 

Show that this has the solution

$$f(r) = C\ln \left( \frac{1}{2}[r + \sqrt{r^2 - C^2}] \right) + K$$

for any $C$ and $K$. Choose $K = -C\ln(C/2)$ so that $f(C) = 0$. What is the physical interpretation of the parameter $C$?

Starting with the definition of $\cosh x$, prove that

$$\cosh^{-1} x = \ln[x + \sqrt{x^2 - 1}].$$

Use this fact to explain why the surface $z = \pm f(r)$ is the same as the surface $r = C\cosh(z/C)$.

a. For $C > 0$, consider the curve $r = C\cosh(z/C)$ in the $wr$-plane. Show that the tangent line through the point $(z_o, r_o)$ on this curve passes through the origin only when $\coth(z_o/C) = z_o/C$.

b. Show that the equation $\coth x = x$ has a unique solution, say $\alpha \approx 1.200$. (Hint: Let $g(x) = \coth x - x$. Show that $g(x)$ is positive for small $x$ and negative for large $x$. Show that $g$ is strictly decreasing for $x > 0$ be calculating $g'$.

c. Show that the tangent lines in part (a) must be of the form $r = \pm z\sinh \alpha$. Hence, regardless of the value of $C$, these lines are tangent to each of the curves $r = C\cosh(z/C)$.

d. From (c) and the convexity of the curves $r = C\cosh(z/C)$ for $C > 0$, deduce that all of these curves are contained in the wedge $r \geq |z|\sinh \alpha$.

e. Conclude that there is no minimal surface joining two coaxial rings of radius $R$, if the rings are separated by a distance of more than $2R/\sinh \alpha < 1.3255R$. 
Probability and the Heat Equation

In this exercise, we shall prove that derive the fundamental solution of the heat equation on probabilistic grounds. Suppose that a particle, starting at the origin has an equal chance of moving to the left or right by a distance $\Delta x$ in time $\Delta t$.

Let $n > 0$ be an integer and $m$ an integer so that $-m \leq m \leq n$ and $n - m$ is even. By computing the number of ways that the particle can move a net distance of $m\Delta x$ in $n$ time intervals $\Delta t$, show that the probability that it is at $x = m\Delta x$ after a time $n\Delta t$ is

$$\frac{n!(\frac{1}{2})^n}{(\frac{1}{2}(n + m))!(\frac{1}{2}(n - m))!}.$$  

(3)

Stirling’s formula tells us that $n! \approx \sqrt{2\pi n}e^{-n}n^n$ for large $n$. We will justify this approximation later. Use the approximation to show that (3) is approximately

$$\frac{2}{\sqrt{2\pi t}} \left[ 1 - \left( \frac{x}{t} \right)^2 \left( \frac{\Delta t}{\Delta x} \right)^2 \right]^{-\frac{1}{2}} \left[ 1 + \left( \frac{x}{t} \right) \left( \frac{\Delta t}{\Delta x} \right) \right]^{-\frac{1}{2} \frac{x}{\Delta x}} \left[ 1 - \left( \frac{x}{t} \right) \left( \frac{\Delta t}{\Delta x} \right) \right]^{\frac{x}{\Delta x}} (\Delta t)^{\frac{1}{2}}.$$  

(4)

We get a well defined density of order $\Delta x$ if $\Delta t$ is proportional to $(\Delta x)^2$, say $(\Delta x^2) = 2k\Delta t$. Divide (4) by $2\Delta x$ and let $\Delta t \to 0$, to obtain the fundamental solution of the heat equation. Hint: Recall that $\lim_{z \to 0} (1 + az)^{1/z} = e^a$.

To complete the problem, we need to justify our use of Stirling’s approximation. Define the Gamma function

$$\Gamma(n + 1) = \int_0^\infty x^n e^{-x} \, dx.$$  

(5)

Use induction to prove that $\Gamma(n + 1) = n!$ for all nonnegative integers $n$. The Gamma function is an extension of the factorial function to real numbers. Make the substitution $x = n + y$ in (5) to obtain

$$\Gamma(n + 1) = n^n e^{-n} \int_{-n}^\infty e^{n \ln(1 + y/n) - y} \, dy.$$  

Use Taylor’s Theorem to expand $\ln(1 + y/n)$ in series, and set $y = v\sqrt{n}$ to see that

$$\Gamma(n + 1) = n^n e^{-n} \int_{-n}^\infty e^{y^2/2n} e^{y^3/3n^2 - y^4/4n^3 + \ldots} \, dy$$

$$= n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty e^{-v^2/2} e^{v^3/3\sqrt{n} - v^4/4n + \ldots} \, dv.$$  

For $n$ large $e^{v^3/3\sqrt{n} - v^4/4n + \ldots} \approx 1$ and $-\sqrt{n} \approx -\infty$. 
