52. Let $T$ be the range of the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(\theta, \psi) = a\langle \cos \theta, \sin \theta, 0 \rangle + b(\cos \psi \cos \theta, \cos \psi \sin \theta, \sin \psi).$$

for $a > b$. Is $T$ a differentiable manifold in $\mathbb{R}^3$? Prove your claim.

53. Let $S$ be the range of the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(\theta, \psi) = \langle \cos \theta \cos \phi \sin \phi, \sin \theta \cos \phi \sin \phi, \cos \theta \sin \theta \cos^2 \phi \rangle.$$

Is $S$ a differentiable manifold in $\mathbb{R}^3$? Prove your claim.

54. Let $H$ be the range of the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(u, v) = \langle u \cos av, u \sin av, v \rangle$$

for $a > 0$. Is $H$ a differentiable manifold in $\mathbb{R}^3$? Prove your claim.

55. Let $C$ be the circle of radius 1 centered at the origin traversed counterclockwise. Let $f(x, y) = 1 + x^3 - y^3$. Evaluate $\int_C f \, ds$.

56. The curve $r : \mathbb{R} \to \mathbb{R}^3$ given by

$$r(t) = \langle t, t^2, t^3 \rangle$$

is the twisted cubic. Graph it. Let $f(x, y, z) = 8x + 12xy + 24z$. Evaluate $\int_C f \, ds$ where $C$ is the portion of the twisted cubic between $t = 0$ and $t = 1$.

57. Let $T$ be the range of the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(\theta, \psi) = a\langle \cos \theta, \sin \theta, 0 \rangle + b(\cos \psi \cos \theta, \cos \psi \sin \theta, \sin \psi).$$

Evaluate $\iint_T 1 \, dS$.

58. The helicoid is the range of the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(u, v) = \langle u \cos av, u \sin av, v \rangle$$

Let $H$ be one ramp of the helicoid, namely the portion

$$H = \{ f(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 2\pi/a \}.$$

Graph the result, and find its area.

59. (20 points) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(r, \theta, \psi) = \langle \cos \theta, \sin \theta, 0 \rangle + r\langle \cos \psi \cos \theta, \cos \psi \sin \theta, \sin \psi \rangle.$$

(a) Explain why for fixed $r < 1$, the range of $f$ is the surface of a torus. Draw a graph.

(b) Explain why the range of $f(r, \theta, \psi)$ for $0 \leq r \leq 1/2$, $0 \leq \theta \leq 2\pi$, $0 \leq \psi \leq 2\pi$ is a solid torus. What are its dimensions? Call this torus $T$.

(c) What is $\partial T$?
(d) Consider the vector field \( g(x, y, z) = (x, y, z) \). Evaluate \( \int \int_{\partial T} g \cdot n \, dS \).

**HINT:** Explain why

\[
g \cdot n \, dS = g \cdot \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \psi} \left| \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \psi} \right| \, d\theta \, d\psi = g \cdot \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \psi} \, d\theta \, d\psi
\]

The triple product is calculable.

(e) Evaluate \( \int \int \int_T \text{div} \, g \, dV \).

**HINT:** Use the Jacobian change of variables given by \( f \), and explain why

\[
dV = \det(Df) \, dr \, d\theta \, d\psi.
\]

Does the divergence theorem hold?

60. Consider the function \( \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

\[
f(u, v) = (u \cos v, u \sin v, v).
\]

Let \( H \) be one ramp of the resulting helicoid, so

\[
H = \{ f(u, v) : 0 \leq u \leq 1, \, 0 \leq v \leq 2\pi \}.
\]

Consider the vector field \( g(x, y, z) = (-y, x, z) \).

(a) Evaluate \( \text{curl} \, g \).

(b) Find the normal \( n \) to \( H \), so that the normal at the origin is \( \langle 0, 1, 0 \rangle \).

(c) Evaluate \( \int \int_H \text{curl} \, g \cdot n \, dS \) using this normal. [Don’t forget the hint in (59d)!]

(d) Explain why \( \partial H = C_1 \cup C_2 \cup C_3 \cup C_4 \) where

\[
C_1 = \{ f(0, v) : 0 \leq v \leq 2\pi \} \quad C_2 = \{ f(u, 2\pi) : 0 \leq u \leq 1 \}
\]

\[
C_3 = \{ f(1, v) : 0 \leq v \leq 2\pi \} \quad C_4 = \{ f(u, 0) : 0 \leq u \leq 1 \}
\]

(e) Evaluate

\[
\int_{\partial H} g \cdot T \, ds = \sum_{i=1}^{4} \int_{C_i} g \cdot T \, ds.
\]

Be sure that the parametrization of \( C_i \) agrees with the choice of \( n \) from (60).

**HINT:** If \( C_i \) is parametrized by \( r(t) \) for \( 0 \leq t \leq a \), then

\[
g \cdot T \, ds = g \cdot \frac{r'(t)}{|r'(t)|} \, |r'(t)| \, dt = g \cdot r'(t) \, dt.
\]

Notice that this is essentially (59d) again.