33. Consider the function 

\[ f(\begin{pmatrix} u \\ v \end{pmatrix}) = \left( \frac{1}{2}(u^2 - v^2) \right). \]

(a) What are the curves of constant \( u \)? What are the curves of constant \( v \)?

(b) Graph \( f \).

(c) Find \( Df \).

(d) Does \( f \) have a (global) inverse?

(e) Where, if anywhere, does \( f \) have a local inverse? Justify your answer.

34. Consider the function \( f : \mathbb{R}^4 \to \mathbb{R}^2 \) given by

\[ f((x_1, x_2), (y_1, y_2)) = \left( \begin{array}{c} x_1^2 - x_2^2 - y_1 \\ 2x_1x_2 - y_2 \end{array} \right). \]

(a) Show \( f(0, 0) = 0 \). [Yes, you need to interpret the zeros as vectors!]

(b) Find \( Df \).

(c) Find \( M \) at the origin. [This is the \( M \) from the Implicit Function Theorem.]

(d) Is there a function \( \rho : \mathbb{R}^2 \to \mathbb{R}^2 \) so that \( f(x, \rho(x)) = 0 \) for all \( x \)? Is there such a function for \( x \) near the origin?

(e) Actually, this problem is trivial. Identify \( \rho \) by inspection.

35. Suppose \( I \) is an \( n \times n \) identity matrix, \( M \) is a \( k \times k \) invertible matrix, \( B \) is an \( k \times n \) matrix, and \( 0 \) is a \( n \times k \) matrix of zeros. Consider the matrix \( A \) in block form

\[ A = \begin{pmatrix} I & 0 \\ B & M \end{pmatrix}. \]

Find the inverse of \( A \).

**Hint:** Look for an inverse in block form.

36. Consider the function \( f : \mathbb{R}^5 \to \mathbb{R}^2 \) given by

\[ f((x_1, x_2, x_3), (y_1, y_2)) = \left( \begin{array}{c} y_2 + e^{y_2} - 2x_1 + x_2x_3 - 1 \\ (1 + y_2)y_1 + x_1x_2 - x_1x_3 \end{array} \right). \]

(a) Show \( f(0, 0) = 0 \).

(b) Find \( Df \).

(c) Find \( M \) at the origin.

(d) Follow the proof of the Implicit Function Theorem (p.59) and define \( F : \mathbb{R}^5 \to \mathbb{R}^5 \) as they do. Find \( DF \). Show it has the block form indicated on p. 59.

(e) Show that \( F \) is invertible near the origin in \( \mathbb{R}^5 \). [Ahhhh- this is why the previous question was asked!]

(f) Conclude from the Inverse Function Theorem that there is a function \( G \) so that \( F \circ G = G \circ F = I \) as maps between portions of \( \mathbb{R}^5 \).

(g) What is \( DG \)? [**Hint:** Have you carefully read p. 54?]
38. Let \( f : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2 \) so that \( f(x, \rho(x)) = 0 \). Use the previous to evaluate \( D\rho(0) \).

37. \( GL(n) \) is the subset of all \( n \times n \) matrices \( M_n \) consisting of all matrices \( A \) so that \( A^{-1} \) exists. Consider \( T : GL(2) \to GL(2) \) given by \( T(A) = A^{-1} \).

(a) Find the derivative \( DT \), where we use the embedding \( GL(2) \subset M_2 \to \mathbb{R}^4 \).

(b) Calculate \( \det DT \); conclude that \( DT \) is locally invertible. Explain why this is trivial.

[\textit{Mathematica} or the equivalent really ought to be used to simplify the algebra.]

38. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix}.
\]

Show that \( f \) is conformal away from the origin, meaning that it preserves angles.

In particular, suppose that \( g : \mathbb{R} \to \mathbb{R}^2 \) and \( h : \mathbb{R} \to \mathbb{R}^2 \) are a pair of smooth curves that intersect at \( g(0) = h(0) = y \neq 0 \). [Recall that \( g : \mathbb{R} \to \mathbb{R}^2 \) is smooth on an interval \( I \) if \( g'(t) \) is continuous on \( I \) and \( |g'(t)| \neq 0 \) on \( I \); see Stewart, \textit{Calculus}, §13.3] Find the angle between the tangent vectors to these curves at this point.

Now consider the images of these curves under \( f \), namely \( f \circ g \) and \( f \circ h \). These curves also intersect at \( f(g(0)) = f(h(0)) = f(y) \). What is the angle between the tangent vectors to these curves at this point? Show this is the same angle found previously.

39. Let \( f : \mathbb{R}^n \to \mathbb{R} \). The Hessian \( \mathcal{H}f \) of \( f \) is the negative of the matrix of the second derivatives of \( \phi \), so that

\[
(\mathcal{H}f)_{ij} = -\frac{\partial^2 f}{\partial x_i \partial x_j}.
\]

Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be given by

\[
f(x_1, x_2, x_3) = x_1 + x_2 + x_3 + 1 - x_1x_2x_3.
\]

Find \( Df \), and find the critical points where \( Df = 0 \). At each critical point, find \( \mathcal{H}f \).

The multi-dimensional analogue of the second derivative test [Stewart, \textit{Calculus}, §4.3] is the Morse Lemma:

**Morse Lemma** Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is infinitely differentiable. Suppose at the point \( x_0 \) we have \( Df(x_0) = 0 \) and that \( \mathcal{H}f(x_0) \) is nonsingular. Then there is a change of coordinates \( y = g^{-1}(x) \) so that

\[
f(x) = f(g(y)) = f(x_0) - \left[y_1^2 + y_2^2 + \cdots + y_\lambda^2\right] + \left[y_{\lambda+1}^2 + \cdots + y_n^2\right]
\]

in a region containing \( x_0 \). The number \( \lambda \) is called the index of the critical point, and is equal to the number of positive eigenvalues of \( \mathcal{H}f(x_0) \).

Use the Morse Lemma to characterize the behavior of \( f \) near the critical points. Are they maxima? Minima? Or something else?

40. Consider the system of equations

\[
\begin{cases}
ux + yv^2 = u - v \\
u^2x^3 + uvy = v - 1
\end{cases}
\]

Is this system uniquely solvable for \( u \) and \( v \) in terms of \( x \) and \( y \) near \( x = 0, y = 0, u = 1, v = 1 \)? Justify your answer.