For problems 1-9, define the necessary operations to make the space into a vector space, or explain why that is impossible.

For those that are vector spaces, determine the dimension of the space. If it is finite-dimensional, exhibit a basis.

1. The space $\mathcal{P}_N$ is defined to be the set of all polynomials with real coefficients of degree less than or equal to $N$.

2. Let $\mathcal{P}_N^*$ be the set of all polynomials $p(x)$ so that $p \in \mathcal{P}_N$ and so that $p(2) = 0$.

3. The space $C[0, 1]$ is defined to be the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$ so that $f$ is continuous on the interval $[0, 1]$.

4. The space $C^k[0, 1]$ is defined to be the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$ so that $f, f', f'', \ldots, f^{(k)}$ are all continuous on the interval $[0, 1]$.

5. Let $\mathcal{N}$ be the set of all functions in $C^2[0, 1]$ so that $f''(x) + f(x) = 0$ for all $x \in [0, 1]$.

6. Let $U$ and $V$ be vector spaces. The space $L(U, V)$ is the space of all linear functions from $U$ to $V$. We say that the function $T : U \rightarrow V$ is linear if $T(\alpha x + \beta y) = \alpha T x + \beta Ty$ for all $x, y \in U$ and for all $\alpha, \beta \in \mathbb{R}$.

7. The space $L^2(0, 1)$ is defined to be the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$ so that $\int_0^1 f^2(x) dx < \infty$.

8. The space $\ell^2$ is defined to be the set of all sequences $\{a_k\}_{k=1}^{\infty}$ so that $\sum_{k=1}^{\infty} a_k^2 < \infty$.

9. The space $M_2$ is defined to be the space of all $2 \times 2$ matrices with real coefficients.

10. Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ be given by $Ty = -(1 - x^2)y'' + 2xy'$. Find the eigenvectors and eigenvalues of $T$.

   **Comments:** We can repeat the problem in $\mathcal{P}_N$ for any positive integer $N$. An eigenvector $p(x)$ normalized so that $p(1) = 1$ is called a Legendre polynomial; these play a fundamental role in the study of solutions of partial differential equations with spherical symmetry.

11. Let $C^\infty[0, 1]$ be the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that all derivatives of $f$ exist. Let $T : C^\infty[0, 1] \rightarrow C^\infty[0, 1]$ be given by $T y = y'$. Find the eigenvectors and eigenvalues of $T$.

   **Comment:** Note that $C^\infty[0, 1]$ is a vector space; this is proven in the same fashion as problem 4.

12. For any $n \times n$ matrix $A$ and any real number $t$, define the matrix $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$. Find $e^{tA}$ when $A = \begin{pmatrix} 3 & 8 \\ 8 & 3 \end{pmatrix}$.

   **Comments:** Theorem 1.8.3 is your friend. The result can be written in terms of hyperbolic trigonometric functions.

13. Let $V = \mathcal{P}_2$. Characterize the space $V^*$. What is $\dim(V^*)$?

14. Let $T : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ be given by $Ty = y'$. Find $T^*$. 
15. Show that any linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ contains a line $\ell$ that is mapped to itself. That is, show that there is a line $\ell$ so that for every $x \in \ell$ we also have $Tx \in \ell$.

**COMMENT:** Eigenvectors.

16. Let $f(x) = x^3$. Prove that $f(x)$ is continuous at every value of $x$. [Use the $\epsilon-\delta$ definition.]

17. The Dirichlet function $\phi(x)$ is defined by

$$
\phi(x) = \begin{cases} 
1 & \text{if } x \text{ is rational}, \\
0 & \text{otherwise}.
\end{cases}
$$

Prove that $\phi(x) = \lim_{m \to \infty} \lim_{n \to \infty} \cos^{2n}(2\pi xn!)$.

18. Show that if $f$ is a differentiable function that $\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$ but that the converse is not true.

19. Consider the function

$$
f(x) = \begin{cases} 
x^2 \sin \left(\frac{1}{x}\right) & x \neq 0, \\
0 & x = 0.
\end{cases}
$$

Use the definition to find $f'(0)$. How often does the tangent line intersect the graph of the function in any interval containing the origin?

20. Prove that if $\phi(x)$ is the Dirichlet function (Problem 17) then the integral $\int_0^1 \phi(x) \, dx$ does not exist.

**COMMENT:** See 2.4.1 and 2.4.2

21. What is Lebesgue's criterion for the (Riemann) integrability of a function? Give a precise statement, defining all terms. Cite an appropriate source.

22. The **Riemann zeta function** is defined by

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
$$
on the interval $(1, \infty)$. Prove that $\zeta(s)$ is continuous on $(1, \infty)$.

**COMMENTS:** The zeta function possesses some interesting properties; for example if we enumerate the prime numbers as $\{p_k\}_{k=1}^{\infty}$, then

$$
\frac{1}{\zeta(s)} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k^s}\right).
$$

It turns out that we can extend the definition of the zeta function to the entire complex plane excluding the point $z = 1$ using the relation

$$
\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k + 1)^{-s}.
$$

An important question in mathematics is the **Riemann Hypothesis** which states that all of the zeros of the zeta function are either the so-called trivial zeros as $s = -2, -4, -6, \ldots$, or lie on the line $\text{Re}(s) = 1/2$. This has been an unsolved question since it was posed in 1859. The Clay Mathematics Institute is currently offering a prize of $1,000,000 for a solution of this problem.

**EXTRA CREDIT:** If you can prove the Riemann Hypothesis not only would you be eligible for the Clay prize, but you would also receive a grade of “A” for this course.
23. Consider the function, called the Hénon map,

\[ f(x) = f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( 1 - \alpha x_1^2 + \frac{x_2}{\beta x_1} \right). \]

Find \( Df \). Does \( f \) have any fixed points?

Consider the sequence of points in the plane \( x, f(x), f(f(x)), f(f(f(x))), \ldots \), called the orbit of \( x \) under the map \( f \). Clearly, for each choice of the initial value of \( x \), we obtain a different orbit. Use Mathematica with the NestList and ListPlot commands to see the orbits of various choices of \( x \). For simplicity, choose \( \alpha = 1.4 \) and \( \beta = 0.3 \), and make sure that you follow the orbit for at least 100,000 iterations. Describe what occurs.

24. Let \( V \) be a vector space. A function

\[ f : V^n \equiv V \times V \times \cdots \times V \rightarrow \mathbb{R} \]

is \( n \)-multilinear if, for any \( x_1, x_2, \ldots, x_n \in V \), any \( y \in V \), any \( \lambda \in \mathbb{R} \), and any \( m \in \{1, 2, \ldots, n\} \) we have

\[ f(x_1, x_2, \ldots, x_m + \lambda y, \ldots, x_n) = f(x_1, x_2, \ldots, x_m, \ldots, x_n) + \lambda f(x_1, x_2, \ldots, y, \ldots, x_n). \]

Prove that any \( n \)-multilinear function \( f : (\mathbb{R}^m)^n \equiv \mathbb{R}^{mn} \rightarrow \mathbb{R} \) is continuous.

Conclude that the function \( \det : M_n \cong \mathbb{R}^{n^2} \rightarrow \mathbb{R} \) is continuous.

25. \( GL(n) \) is the subset of \( M_n \) consisting of all matrices \( A \) so that \( A^{-1} \) exists.

Let \( T : GL(2) \rightarrow GL(2) \) be given by \( T(A) = A^{-1} \). Find the derivative \( DT \), where we use the embedding \( GL(2) \subset M_2 \hookrightarrow \mathbb{R}^4 \).

26. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by

\[ f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} x_2^2 - x_1^2 \\ 2x_1 x_2 \end{array} \right). \]

Show that \( f \) is conformal, meaning that it preserves angles.

In particular, suppose that \( g : \mathbb{R} \rightarrow \mathbb{R}^2 \) and \( h : \mathbb{R} \rightarrow \mathbb{R}^2 \) are a pair of \( C^1 \) curves that intersect at \( g(0) = h(0) = y \). Find the angle between the tangent vectors to these curves at this point.

Now consider the images of these curves under \( f \), namely \( f \circ g \) and \( f \circ h \). These curves also intersect at \( f(g(0)) = f(h(0)) = f(y) \). What is the angle between the tangent vectors to these curves at this point? Show that this is the same angle found previously.

27. Let \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \). The Hessian \( \mathcal{H}\phi \) of \( \phi \) is the negative of the matrix of the second derivatives of \( \phi \), so that

\[ (\mathcal{H}\phi)_{ij} = -\frac{\partial^2 f}{\partial x_i \partial x_j}. \]

Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be given by

\[ f(x_1, x_2, x_3) = x_1 + x_2 + x_3 + 1 - x_1 x_2 x_3. \]

Find \( Df \), and find the critical points where \( Df = 0 \). Find \( \mathcal{H}f \), then use the Morse Lemma to determine the behavior of \( f \) near its critical points.
**Morse Lemma** Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is infinitely differentiable. Suppose at the point \( x_0 \) we have \( Df(x_0) = 0 \) and that \( \mathcal{H}f(x_0) \) is nonsingular. Then there is a change of coordinates \( y = g^{-1}(x) \) so that

\[
f(x) = f(g(y)) = f(x_0) - \left[ y_1^2 + y_2^2 + \cdots + y_3^2 \right] + \left[ y_{n+1}^2 + \cdots + y_n^2 \right]
\]

in a region containing \( x_0 \). The number \( \lambda \) is called the index of the critical point, and is equal to the number of positive eigenvalues of \( \mathcal{H}f(x_0) \).

28. Consider the system of equations

\[
\begin{align*}
ux + yv^2 &= y \\
u^2x^3 + uv &= 1
\end{align*}
\]

Is this system uniquely solvable for \( u \) and \( v \) in terms of \( x \) and \( y \) near \( x = 0, y = 0, u = -1, v = -1 \).

29. Let \( X = \{1,2,3\} \). How many subsets of \( X \) are there? How many collections of subsets of \( X \) are there? How many different topologies of \( X \) are there? Enumerate them.

30. Consider the set \( \mathbb{C}^* = \mathbb{C} \cup \{\infty\} \). To define a topology on \( \mathbb{C}^* \), we define a function \( f \) from the sphere to \( \mathbb{C}^* \) via the following geometry

\[
\begin{align*}
S^2 & \quad \text{S} \\
x^* & \quad \text{X} \\
f(x)^* & \quad \text{F}
\end{align*}
\]

where the north pole of the sphere is mapped to the point \( \infty \). We then say that a set \( U \subseteq \mathbb{C}^* \) is open if and only if its preimage \( f^{-1}(U) \) is open in \( S^2 \). Prove that \( \mathbb{C}^* \) is compact.

31. Let \( (X, \rho) \) be a metric space, and define

\[
\tilde{\rho}(x,y) = \frac{\rho(x,y)}{1 + \rho(x,y)}.
\]

Show that \( (X, \tilde{\rho}) \) is also a metric space.

32. Suppose that \( (X_1, T_1) \) and \( (X_2, T_2) \) are topological spaces. A bijection \( f : X_1 \to X_2 \) is a homeomorphism if both \( f \) and \( f^{-1} \) are continuous. Consider the list of subsets of \( \mathbb{R}^2 \) given by the figures of the letters

\[
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z.
\]

Sort this list into collections of homeomorphic figures.
33. Consider the space $C^k[0,1]$ of all functions from $[0,1]$ to $\mathbb{R}$ whose first $k$ derivatives are all continuous. For any two functions $f, g \in C^k[0,1]$, define

$$\rho_k(f, g) = \sup |f(x) - g(x)| + \sup |f'(x) + g'(x)| + \cdots + \sup |f^{(k)}(x) - g^{(k)}(x)|.$$ 

Show that $(C^k[0,1], \rho_k)$ is a metric space.

34. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The closure $\bar{A}$ of $A$ is the set

$$\bar{A} = \bigcap\{F : F \supseteq A, F \text{ closed}\}.$$ 

(a) Prove that if $A$ is closed then $\bar{A} = A$.

(b) Prove that $x \in \bar{A}$ if and only if every open set containing $x$ has nonempty intersection with $A$.

(c) For $X = \mathbb{R}^k$ for some $k$, prove that $x \in \bar{A}$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ so that $x_n \to x$, (using the usual $\epsilon-N$ definition of the convergence of a sequence.)

35. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The interior $\overset{\circ}{A}$ of $A$ is the set

$$\overset{\circ}{A} = \bigcup\{U : U \subseteq A, U \text{ open}\}.$$ 

For each of the following choices of $A$, find $\bar{A}$ and $\overset{\circ}{A}$.

(a) $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$

(b) $A = \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}$.

(c) $A = \{x \in \mathbb{R} : x = 1/n \text{ for some } n \in \mathbb{N}\}$

(d) $A = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$

36. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The boundary $\partial A$ of the set $A$ is the set

$$\partial A = \bar{A} \cap (X \setminus A).$$

Show that, if $A$ is open then $\partial A = \bar{A} \setminus A$.

37. Find a parameterization of a torus with inner radius $a$ and outer radius $b$. Show that the torus is a 2-manifold by verifying the conditions of Definition 5.1.2.

38. Consider the vector field $\mathbf{v} = \langle x, y, z \rangle$. Verify that the divergence theorem holds for this vector field over a torus with inner radius 1 and outer radius 2.

[Be sure to explicitly evaluate both integrals.]

39. According to the Biot-Savart law, an electrical current $I$ flowing upward in a wire along the $z$-axis produces a magnetic field $\mathbf{H}$ of the form

$$\mathbf{H} = \frac{2I}{|r|^3} \mathbf{k} \times \mathbf{p}$$

where $\mathbf{p} = \langle x, y, 0 \rangle$. Show that if $C$ is any smooth simple closed curve enclosing the $z$-axis then $\int_C \mathbf{H} \cdot d\mathbf{r} = 4\pi I$, while if $C$ does not enclose the $z$-axis, then $\int_C \mathbf{H} \cdot d\mathbf{r} = 0$.

40. Consider the square where the sides are identified as shown in the figure below. Is the result a manifold? If so, explain how to parameterize it.
41. Let \( \mathbf{v}_1 = \langle 1, 3, 5 \rangle \) and \( \mathbf{v}_2 = \langle -1, -2, -1 \rangle \). Let \( P \) be the parallelepiped spanned by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Calculate \( dx_1 \wedge dx_2(P) \), \( dx_1 \wedge dx_3(P) \) and \( dx_2 \wedge dx_3(P) \).

42. Let \( \mathbf{v}_1 = \langle 1, 3, 5, 7, 9 \rangle \), \( \mathbf{v}_2 = \langle -1, -2, -1, -1, -2 \rangle \), and \( \mathbf{v}_3 = \langle 3, 1, 4, 1, 5 \rangle \). Let \( P \) be the parallelepiped spanned by \( \mathbf{v}_1 \), \( \mathbf{v}_2 \), and \( \mathbf{v}_3 \). What are the values of all of the elementary 3-forms acting on \( P \)?

43. Prove that \( \bigwedge^k(\mathbb{R}^n) \) is a vector space.

44. In \( \mathbb{R}^n \), show that \( dx_i \wedge dx_j = -dx_j \wedge dx_i \) for any \( i, j \).

45. In \( \mathbb{R}^n \), show that for any one forms \( d\omega \) and \( d\tau \) that \( d\omega \wedge d\tau = -d\tau \wedge d\omega \).

46. Use the definition of the wedge product in \( \mathbb{R}^3 \) to show that \( (dx_1 \wedge dx_2) \wedge dx_3 = dx_1 \wedge (dx_2 \wedge dx_3) \).

47. Prove that for any differential \( k \)-form \( \omega \), we have \( d(d\omega) = 0 \).
Choose any one of the following assignments. You will then prepare a short paper (8-15 pages) that describes both the material that you read as well as your answers to the given questions. The paper should be well-written and follow all of the usual rules for composition, including spelling, grammar, and the use of references as appropriate.

You are also required to prepare a short (15-minute) presentation that will be made to the class and interested members of the Towson community that describes both the material that you read as well as your answers to the given questions.

The written paper is due before the final exam period for this course; in particular it is due at or before 12:30 on Tuesday, December 12. Late work will not be accepted without an extremly compelling reason.

The oral presentations will be scheduled for the last week of class, before the final exam period.

You may complete this assignment individually, or you make work with one partner. Please inform the instructor if you will be working with a partner, and if so, who your partner is.

1. Choice #1. Read Chapter 11, sections 11.1, 11.2, and 11.3. Answer homework questions 1, 2, and 3. In addition, answer the following:
   - Consider the group $G = \{1, a, b, ab\}$ where $a^2 = b^2 = 1$, and $ab = ba$. For each $g \in G$, define $T_g : G \rightarrow G$ by $T_g x = gx$. Explain why we can identify $T_g$ with an element of $S_4$, the group of permutations on four elements. For each $g \in G$, find the corresponding element of $S_4$.
   - Choose unit vectors $e_g \in \mathbb{R}^4$ for each $g \in G$. Then for each $g \in G$, define $\rho(g) \in GL(\mathbb{R}^4)$ so that $\rho(g)e_x = e_{gx}$ for all $x \in G$. Write out the matrices $\rho(g)$ for all $g \in G$.
   - Show that the mapping $\rho : G \rightarrow GL(\mathbb{R}^4)$ is a representation.
   - Let $T$ be any fixed invertible $4 \times 4$ matrix. Prove that $\tau : G \rightarrow GL(\mathbb{R}^4)$ given by $\tau(g) = T\rho(g)T^{-1}$ is also a representation.

2. Choice #2. Read Chapter 14, skipping section 14.7. Answer homework questions 1 and 4. In addition, answer the following:
   - Using Mathematica or otherwise, graph some representative solutions to the wave equation. Use the graphs to explain why it is called the wave equation.
   - Solve the Cauchy problem for the heat equation; that is find the function $u(x, t)$ that satisfies
     \[
     \begin{align*}
     u_t &= cu_{xx} & -\infty < x < \infty, \ t > 0; \\
     u(x, 0) &= f(x) & -\infty < x < \infty.
     \end{align*}
     \]
     Use the solution to show that the effect of a source of heat is immediately felt infinitely far away from the source, and contrast this behavior with that of the wave equation.


4. Choice #4. Read Chapter 16, skipping section 16.5. Answer questions 1, 2, 3, 4.

5. Choice #5. Read the provided material on Game Theory. Solve “The Price is Right!” game.