Final Examination
Math 284
May 22, 2000

Name ______________________________

1. Evaluate
   a. \( \int \frac{dx}{x^3 + x} \)
   b. \( \int_0^2 \frac{8}{\sqrt{4 - x^2}} \, dx \)

2. The average speed of molecules in an ideal gas is
   \[ \bar{v} = \frac{4}{\sqrt{\pi}} k^{3/2} \int_0^\infty v^3 e^{-kv^2} \, dv \]
   where \( k = \frac{M}{2RT} \), and where \( M \) is the molecular weight of the gas, \( R \) is the gas constant, \( T \) is the temperature, and \( v \) is the molecular speed. Show that
   \[ \bar{v} = \frac{\sqrt{8RT}}{\pi M} \]

3. What is Simpson’s rule? Describe it, and explain its associated error. What happens to the error if the number of subintervals is increased by a factor of 10? What is the advantage of Simpson’s rule over other methods?

4. The arc of the parabola \( y = x^2 \) from (1,1) to (2,4) is revolved about the y-axis. Find the area of the resulting surface.

5. Find the (exact) centroid of the region bounded by \( y = \sin \frac{\pi x}{2} \), \( y = 0 \), \( x = 0 \), and \( x = 2 \).

6. Find the volume of the solid obtained by revolving the region bounded by \( y = x^3 \), \( y = 8 \), and \( x = 0 \) about the y-axis.

7. Scientists can determine the age of ancient objects by a method called radiocarbon dating. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, \(^{14}\text{C}\), with a half-life of 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates \(^{14}\text{C}\) through food chains. When a plant or animal dies, it stops replacing its carbon, and the amount of \(^{14}\text{C}\) begins to decrease through radioactive decay. Therefore the level of radioactivity also must decay. A parchment fragment is discovered that has 74% of the radioactivity that plant material does now. Estimate the age of the parchment.

8. Use Euler’s method with a step size of 0.2 to estimate \( y(1) \), where \( y(t) \) is the solution to the initial value problem \( y'(t) = t + y^2 \), \( y(0) = -1 \).
9. What is the direction field for a differential equation? Sketch the direction field for \( y' = y - t \), and draw in some representative solutions.

10. What is the definition of \( \sum_{n=1}^{\infty} a_n \)?

11. Give three different tests that can be used to test for the convergence of a series, and explain how each is used.

12. Suppose that \( f(x) = T_n(x) + R_n(x) \) where \( T_n(x) \) is the \( n \)th degree Taylor polynomial for \( f(x) \) at \( x = a \). Write down an expression for \( T_n(x) \) and for \( R_n(x) \).

13. Determine if the series converges absolutely and/or conditionally. Explain why your answer is correct.
   
   a. \( \sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{3n^3 + 4n^2 + 2} \),
   
   b. \( \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^4 + 1} \),
   
   c. \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \).

14. Evaluate \( \int_0^{1/2} \frac{dx}{1 + x^5} \) accurately to \( 10^{-5} \) with the aid of a power series. Demonstrate why your answer has the required degree of accuracy.

15. Let \( f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \), and find the radius of convergence of \( f(x) \). Show that \( f'(x) + 2xf(x) = 0 \).

16. In Einstein’s special theory of relativity, the mass of an object moving with speed \( v \) is

\[
m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}
\]

where \( m_0 \) is the mass of the object at rest, and \( c \) is the speed of light. The kinetic energy of the object is the difference between its total energy and its rest energy, namely

\[
K = mc^2 - m_0c^2.
\]

Show that if \( v \) is very small relative to \( c \), then

\[
K \approx \frac{1}{2} m_0 v^2.
\]