Consider the two functions $f(x) = 1 - x$ and $g(x) = \frac{1}{x}$. These functions can be composed in two different ways, as $f(g(x))$ and $g(f(x))$. We can go further, and compose these new functions with each other or with the original functions. Keep composing these functions with the new ones as they are generated, and simplify the functions you obtain. (Don’t forget to compose functions with themselves, like $f(f(x))$.) You might think that by doing so one would obtain an infinite family of functions. Surprisingly, only a finite number of new functions can be generated in this fashion, even though there may be infinitely many ways of composing $f$ and $g$ to get the same function. Remember that two different looking expressions may represent the same function.

(a) How many distinct functions are there, including $f$ and $g$ themselves?
(b) List them.
(c) How is each composed from $f$ and $g$?
(d) How do you know that these are all that there are?
(e) What is the domain and range of each function?
(f) For what real numbers are all of these functions simultaneously defined?

Rules:
(1) The assignment is due at the beginning of class on Wednesday, September 22, 1999.
(2) This project is to be completed by teams of 2 students. You may not discuss this project with other members of class, or with anyone outside of class. You may (only) discuss the project with the instructor.
(3) Aside from the restrictions in (2) above, the use of outside references is acceptable, and indeed encouraged. However, all outside references need to be properly acknowledged.
(4) Answers should be a well-written paper that describes the problem and the solution. All of the usual rules of English grammar and composition apply.
(5) Papers need to be neat, clean, and paper-clipped or stapled. They do not need to be typed or written in ink, but they must be legible and easily readable.
(6) Copying the work of another student or portions of a published work constitutes plagiarism. Plagiarism or any other form of academic dishonesty may be reported to the Dean for action.

This project is a modification of a project from M. Cohen, et. al., *Student Research Projects in Calculus*, The Mathematical Association of America, 1991.