

A New Point of View on General Kaluza-Klein Theories

Aurel BEJANCU^{*)}

Department of Mathematics, Kuwait University, Safat 5969, Kuwait

(Received April 15, 2012; Revised June 20, 2012)

The general Kaluza-Klein theories are physical theories in which both the “cylinder condition” and the “compactification condition” from the classical Kaluza-Klein theory are not necessarily satisfied. Our study is developed on a general Kaluza-Klein space $(\bar{M} = M \times K, \bar{g})$, whose tangent bundle $T\bar{M}$ splits into horizontal and vertical distributions $H\bar{M}$ and $V\bar{M}$, respectively. The main tool in our new point of view is what we call the Riemannian horizontal connection ∇ on $H\bar{M}$, which plays in a general Kaluza-Klein theory, the same role as the Levi-Civita connection on the spacetime M in the classical Kaluza-Klein theory. This connection enables us to classify the geodesics of (\bar{M}, \bar{g}) , to define the horizontal Einstein gravitational tensor field, and to write down in a covariant form, the field equations on (\bar{M}, \bar{g}) . In particular, we apply the study to both the theory of Einstein-Bergmann spaces and the theory of general Kaluza-Klein spaces with bundle-like metric.

Subject Index: 012, 013, 180, 450, 453

Introduction

In 1919, Kaluza had the brilliant idea to use a 5-dimensional manifold to unify the Einstein theory of general relativity with the Maxwell theory of electromagnetism. His paper¹⁰⁾ was published after a delay of two years, and since then several studies have been developed with respect to both the physical and mathematical point of views. Kaluza’s achievement was possible under a strong condition, which assumes that all the local components of the pseudo-Riemannian metric on the 5-dimensional manifold do not depend on the fifth coordinate. This is known in literature as the “cylinder condition”. Later on, Klein¹²⁾ added the “condition of compactification”, which consists in the assumption that the space is closed by a very small circle in the direction of the fifth dimension.

In 1938, Einstein and Bergmann⁷⁾ presented the first generalization of the Kaluza-Klein theory. This consists in the fact that the local components of the 4-dimensional pseudo-Riemannian metric are periodic functions of the fifth coordinate. However, the electromagnetic potentials are still independent of the fifth coordinate, that is, the cylinder condition is only partially satisfied. It is noteworthy that in 7), for the first time in a general Kaluza-Klein theory, it is defined a covariant differentiation on a vector bundle of rank 4 (in the modern terminology) over the 5-dimensional manifold.

A well known generalization of the 5-dimensional Kaluza-Klein theory is the so-called “space-time-matter theory” (cf. Wesson²⁸⁾). According to this theory, the matter in the 4-dimensional spacetime is considered to be a manifestation of the fifth dimension. There have been found several cosmological solutions in which both the

^{*)} E-mail: aurel.bejancu@ku.edu.kw

cylinder condition and the compactification of the fifth dimension were removed (cf. Ponce de Leon,²⁰ Chatterjee,³ Liu and Wesson,¹⁵,¹⁶ Billyard and Wesson,² Ponce de Leon and Wesson²²). An excellent survey on space-time-matter theories can be found in Overduin and Wesson.¹⁹)

There are some other generalizations of the Kaluza-Klein theory to the $(4+k)$ -dimensional spaces with respect to non-abelian gauge groups (cf. Kerner,¹¹ Cho,⁴ Cho and Freund,⁵ Salam and Strathdee,²⁶ Coquereaux and Jadczyk⁶). We think that the new point of view presented here on the 5-dimensional case with abelian group $U(1)$ can also be considered for the study of such theories.

Next, we describe the new point of view we want to present on general 5-dimensional Kaluza-Klein theories. The study is developed in a general Kaluza-Klein space $(\bar{M} = M \times K, \bar{g})$, where M and K are 4- and 1-dimensional manifolds, respectively, and \bar{g} is a pseudo-Riemannian metric on \bar{M} subject to some conditions. Consider \bar{M} as a trivial bundle over M , and denote by $V\bar{M}$ the vertical bundle on \bar{M} , which is tangent to the foliation whose leaves are $\{x\} \times K$, $x \in M$. The gauge transformations on \bar{M} are determined by the horizontal distribution $H\bar{M}$, which is supposed to be a Lorentz vector bundle that is complementary orthogonal to $V\bar{M}$ in $T\bar{M}$ with respect to \bar{g} . This geometrical framework enables us to construct what we call the Riemannian horizontal connection on \bar{M} , which is the main object in our approach. More precisely, this connection is a linear connection on $H\bar{M}$ which is a metric connection with prescribed torsion tensor field. It is noteworthy that the Riemannian horizontal connection on $H\bar{M}$ plays a role in our general Kaluza-Klein theory that is similar to the role of Levi-Civita connection in the 4-dimensional spacetime. By using this connection we succeed to write down in a covariant form, both the equations of motion and the field equations in (\bar{M}, \bar{g}) . New geometrical objects like: horizontal electromagnetic tensor field, horizontal electromagnetic energy-momentum tensor field and horizontal Einstein tensor field, have an important role in the study. In particular, we apply the theory developed on (\bar{M}, \bar{g}) to the Einstein-Bergmann spaces, and to the general Kaluza-Klein spaces with bundle-like metrics.

Now, we outline the content of the paper. In the first section, we present the general Kaluza-Klein space (\bar{M}, \bar{g}) and introduce the adapted coordinate systems and adapted frame fields on (\bar{M}, \bar{g}) . Then, in §2, we shortly present the theory of horizontal tensor fields on (\bar{M}, \bar{g}) . In particular, we define the horizontal electromagnetic tensor field $F = (F_{\alpha\beta})$ (cf. (2.4a)) and two important horizontal tensor fields $B = (B_\alpha)$ (cf. (2.4b)) and $D = (D_{\alpha\beta})$ (cf. (2.12)). More about adapted tensor fields on foliated manifolds can be found in the book of Bejancu and Farran.¹ Next, in §3, we construct the Riemannian horizontal connection ∇ on $H\bar{M}$ (cf. Theorem 3.1) and relate it with the Levi-Civita connection $\bar{\nabla}$ on (\bar{M}, \bar{g}) . Also, we express the local coefficients of $\bar{\nabla}$ with respect to an adapted frame field, by means of local coefficients of ∇ and the horizontal tensor fields defined in the previous section (cf. Theorem 3.2). Then, in §4, we write down in a covariant form the equations of motion in (\bar{M}, \bar{g}) (cf. Theorem 4.1) and study two categories of geodesics in (\bar{M}, \bar{g}) : the horizontal and non-horizontal geodesics. It is noteworthy that *horizontal geodesics of (\bar{M}, \bar{g}) must be autoparallels of the Riemannian horizontal connection on*

$H\bar{M}$ (cf. Corollary 4.2). Projectable geodesics of (\bar{M}, \bar{g}) will define horizontal and non-horizontal induced motions on the base manifold M . In §§5 and 6, we apply our method of study to Einstein-Bergmann spaces and general Kaluza-Klein spaces with bundle-like metrics, respectively. In particular, we show that in the case of the classical Kaluza-Klein space (\bar{M}, \bar{g}) , any motion of the spacetime (M, g) is a horizontal induced motion and vice versa (cf. Corollary 5.4), and the solutions of the Lorentz force equations on (M, g) are the projections of some non-horizontal geodesics of (\bar{M}, \bar{g}) (cf. Corollary 5.5). Also, in the case of general Kaluza-Klein spaces with bundle-like metric, we find a geometrical condition on non-horizontal geodesics of (\bar{M}, \bar{g}) in order to become a classical Kaluza-Klein space (cf. Theorem 6.6). Next, in §7, we prove some Bianchi identities for the Riemannian horizontal connection ∇ (cf. Theorem 7.1), and state the main properties of the curvature tensor field of ∇ (cf. Theorem 7.2). Then we define the horizontal Ricci tensor for ∇ , which under the constraint (7.26) becomes a symmetric horizontal tensor field of type $(0, 2)$. The horizontal scalar curvature of (\bar{M}, \bar{g}) is locally given by a formula (cf. (7.25)) that is similar to the well known formula from Riemannian geometry. The main difference consists in the fact that the local components in the right part of (7.25) are taken with respect to the adapted frame fields. In case (\bar{M}, \bar{g}) is a relativistic general Kaluza-Klein space, we prove (7.33) which has a great role in the next section. In §8, we define the Einstein horizontal gravitational tensor field and prove that it is a symmetric horizontal tensor field whose horizontal divergence vanishes identically on \bar{M} (cf. Theorem 8.1). Then, in §9, we express both the Ricci tensor and the scalar curvature of (\bar{M}, \bar{g}) in terms of the horizontal Ricci tensor, horizontal scalar curvature and the local components of the horizontal tensor fields F , D and B (cf. Theorems 9.1 and 9.2). Finally, in §10, we define the horizontal electromagnetic energy-momentum tensor field (cf. (10.4)) and express the local components of the Einstein gravitational tensor field of a relativistic general Kaluza-Klein space (\bar{M}, \bar{g}) (cf. Proposition 10.1). This enables us to write, in a covariant form, the Einstein equations on (\bar{M}, \bar{g}) (cf. Theorem 10.1). The horizontal Einstein equation (10.13) represents the generalization of the field equations that unify the Einstein theory of general relativity with the Maxwell theory of electromagnetism from the classical Kaluza-Klein theory. Finally, in §11 we present a comparison between our approach and what is known so far in literature. In particular, we compare our new point of view on general Kaluza-Klein theories with the approach of Wesson and his collaborators (Ref. 28)) on space-time-matter theory. We close the paper with Conclusions.

§1. The general Kaluza-Klein space

In this section we present the geometric structure of a general Kaluza-Klein space. This space is a generalization of the classical Kaluza-Klein space, in the sense that both the “cylinder condition” on the pseudo-Riemannian metric and the “compactification condition” on the fifth dimension are not necessarily satisfied. Such a generalization was intensively studied in the last two decades under the name of space-time-matter theory (cf. Wesson²⁸). The purpose of this theory is to

show how matter in the 4-dimensional spacetime is induced by the geometry of the ambient 5-dimensional general Kaluza-Klein space. We note that the whole study in the paper is developed by using the adapted coordinate systems and the adapted frame fields, which we introduce in this section.

Let M and K be two manifolds of dimensions four and one, respectively. Consider $\bar{M} = M \times K$ as a trivial bundle over M with respect to the projection π on the first factor. Then, any coordinate system (x^α) on M defines a fibred coordinate system (x^α, x^4) on \bar{M} , where x^4 is the fibre coordinate.

Two such coordinate systems $(\tilde{x}^\alpha, \tilde{x}^4)$ and (x^α, x^4) are related by

$$(a) \tilde{x}^\alpha = \tilde{x}^\alpha(x^0, x^1, x^2, x^3), \quad (b) \tilde{x}^4 = \tilde{x}^4(x^0, x^1, x^2, x^3, x^4). \quad (1.1)$$

Throughout the paper, we use the ranges of indices: $\alpha, \beta, \gamma, \dots \in \{0, 1, 2, 3\}$, $i, j, k, \dots \in \{0, 1, 2, 3, 4\}$. Also, we use Einstein's convention, that is, repeated indices with one upper index and one lower index denotes summation over their range. For any vector bundle E over \bar{M} , we denote by $\Gamma(E)$ the $\mathcal{F}(\bar{M})$ -module of smooth sections of E , where $\mathcal{F}(\bar{M})$ is the algebra of smooth functions on \bar{M} .

Next, from (1.1) we deduce that the transformations of the natural frame fields on \bar{M} have the form

$$(a) \frac{\partial}{\partial x^\alpha} = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\gamma} + \frac{\partial \tilde{x}^4}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^4}, \quad (b) \frac{\partial}{\partial x^4} = \frac{\partial \tilde{x}^4}{\partial x^4} \frac{\partial}{\partial \tilde{x}^4}. \quad (1.2)$$

Then, due to (1.2b), we have a line vector bundle $V\bar{M}$ over \bar{M} that is locally spanned by $\partial/\partial x^4$. We call $V\bar{M}$ the *vertical distribution* on \bar{M} . Now, suppose that on \bar{M} there exists a complementary distribution $H\bar{M}$ to $V\bar{M}$ in the tangent bundle $T\bar{M}$ of \bar{M} , which we call the *horizontal distribution* on \bar{M} . Hence we have the direct decomposition

$$T\bar{M} = H\bar{M} \oplus V\bar{M}. \quad (1.3)$$

Also, we suppose that there exists on \bar{M} a pseudo-Riemannian metric \bar{g} satisfying the following conditions:

- (i) The induced metric on $H\bar{M}$ by \bar{g} is a Lorentz metric, which means that it is nondegenerate of signature $(+ + + -)$.
- (ii) $H\bar{M}$ and $V\bar{M}$ are orthogonal vector bundles with respect to \bar{g} .

By using \bar{g} , we construct around any point $\bar{x} \in \bar{M}$ a special fibred coordinate system, as follows. Let $(\bar{x}^\alpha, \bar{x}^4)$ be a fibred coordinate system and ξ be a local unit vertical vector field, that is, we have

$$(a) \xi = f(\bar{x}^\alpha, \bar{x}^4) \frac{\partial}{\partial \bar{x}^4}, \quad (b) \bar{g}(\xi, \xi) = \varepsilon, \quad (1.4)$$

where $\varepsilon = +1$ or $\varepsilon = -1$, according as $V\bar{M}$ is spacelike or timelike vector bundle, respectively. Then, we define

$$x^\alpha = \bar{x}^\alpha, \quad x^4 = \int_a^{\bar{x}^4} \frac{1}{f(\bar{x}^\alpha, t)} dt, \quad a \in \mathbb{R},$$

and obtain a fibred coordinate system (x^α, x^4) such that

$$\xi = \frac{\partial}{\partial x^4}. \tag{1.5}$$

Thus, by (1.5) and (1.4b), we can state that *around each point $\bar{x} \in \bar{M}$ there exists a fibred coordinate system (x^α, x^4) on \bar{M} such that*

$$\bar{g}_{44} = \bar{g} \left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4} \right) = \varepsilon. \tag{1.6}$$

We call (x^α, x^4) satisfying (1.6) an *adapted coordinate system* on \bar{M} .

From now on, we suppose that the transformations of coordinates on \bar{M} have the particular form

$$(a) \tilde{x}^\alpha = \tilde{x}^\alpha(x^0, x^1, x^2, x^3), \quad (b) \tilde{x}^4 = x^4 + h(x^0, x^1, x^2, x^3), \tag{1.7}$$

where h is a smooth function locally defined on M . These transformations were named by Einstein and Bergmann⁷⁾ the *four- and cut-transformations*. In this case, (1.2) becomes

$$(a) \frac{\partial}{\partial x^\alpha} = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\gamma} + \frac{\partial h}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^4}, \quad (b) \frac{\partial}{\partial x^4} = \frac{\partial}{\partial \tilde{x}^4}. \tag{1.8}$$

Now, we consider an adapted coordinate system (x^α, x^4) and a local frame field $\{E_\alpha, \partial/\partial x^4\}$ on \bar{M} , where $E_\alpha \in \Gamma(H\bar{M})$ for all $\alpha \in \{0, 1, 2, 3\}$. Then express $\partial/\partial x^\alpha$ as follows:

$$\frac{\partial}{\partial x^\alpha} = A_\alpha^\gamma E_\gamma + A_\alpha \frac{\partial}{\partial x^4}, \tag{1.9}$$

where A_α^γ and A_α are smooth functions locally defined on \bar{M} . As the transition matrix from $\{E_\alpha, \partial/\partial x^4\}$ to the natural frame field $\{\partial/\partial x^\alpha, \partial/\partial x^4\}$ is

$$\begin{bmatrix} A_\alpha^\gamma & 0 \\ A_\alpha & 1 \end{bmatrix},$$

we deduce that the 4×4 matrix $[A_\alpha^\gamma]$ is nonsingular. Hence, the vector fields

$$\frac{\delta}{\delta x^\alpha} = A_\alpha^\gamma E_\gamma,$$

also represent locally the horizontal distribution. Moreover, from (1.9) we deduce that

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A_\alpha \frac{\partial}{\partial x^4}. \tag{1.10}$$

We call $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ the *adapted frame field* corresponding to the adapted coordinate system (x^α, x^4) . If $(\tilde{x}^\alpha, \tilde{x}^4)$ is another adapted coordinate system, by using (1.10) for both coordinate systems, and (1.8), we deduce that

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\delta}{\delta \tilde{x}^\gamma} + \left\{ \tilde{A}_\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} + \frac{\partial h}{\partial x^\alpha} - A_\alpha \right\} \frac{\partial}{\partial \tilde{x}^4}.$$

Hence, we have

$$A_\alpha = \tilde{A}_\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} + \frac{\partial h}{\partial x^\alpha}, \tag{1.11}$$

and

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\delta}{\delta \tilde{x}^\gamma}. \tag{1.12}$$

We should note that (1.11) looks like the gauge transformations of the electromagnetic vector potentials. The difference is that A_α and \tilde{A}_γ are not supposed in (1.11) to be independent of the fifth coordinate.

Remark 1.1 From now on, all the calculations around any point of \bar{M} will be performed by using both the adapted coordinate system and the adapted frame field. ■

Next, we denote by g the Lorentz metric induced by \bar{g} on $H\bar{M}$. Then we put

$$g_{\alpha\beta} = g \left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right) = \bar{g} \left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right). \tag{1.13}$$

Also, taking into account the condition (ii), we obtain

$$\bar{g} \left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4} \right) = 0. \tag{1.14}$$

Hence, by (1.6), (1.13) and (1.14), we deduce that the matrix of the pseudo-Riemannian metric \bar{g} with respect to the adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ is expressed as follows:

$$\begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & \varepsilon \end{bmatrix}. \tag{1.15}$$

On the other hand, by using (1.10) into (1.13) and (1.14), and taking into account (1.6), we obtain

$$(a) \bar{g} \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = g_{\alpha\beta} + \varepsilon A_\alpha A_\beta, \quad (b) \bar{g} \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^4} \right) = \varepsilon A_\alpha. \tag{1.16}$$

Thus, the matrix of \bar{g} with respect to the natural frame field $\{\partial/\partial x^\alpha, \partial/\partial x^4\}$ is much more complicated than the one in (1.15). More precisely, it has the form

$$\begin{bmatrix} g_{\alpha\beta} + \varepsilon A_\alpha A_\beta & \varepsilon A_\alpha \\ \varepsilon A_\beta & \varepsilon \end{bmatrix}. \tag{1.17}$$

Now, we define the local 1-form on \bar{M} :

$$\delta x^4 = dx^4 + A_\alpha dx^\alpha. \tag{1.18}$$

Then $\{dx^\alpha, \delta x^4\}$ is an *adapted coframe field* on \bar{M} . Thus, the matrix representations of \bar{g} in (1.15) and (1.17) are equivalent to

$$\bar{g} = g_{\alpha\beta} dx^\alpha dx^\beta + \varepsilon (\delta x^4)^2, \tag{1.19}$$

and

$$\bar{g} = (g_{\alpha\beta} + \varepsilon A_\alpha A_\beta) dx^\alpha dx^\beta + 2\varepsilon A_\alpha dx^\alpha dx^4 + \varepsilon(dx^4)^2, \tag{1.20}$$

respectively.

Remark 1.2 All the functions $g_{\alpha\beta}$ and A_α from (1.17) are defined on the domain of an adapted coordinate system on \bar{M} , and therefore, they are not supposed to depend only on the spacetime coordinates (x^α) . Thus we remove the cylinder condition from classical Kaluza-Klein theories. Also, we do not require that K is a circle. ■

In particular, we consider A_α as the electromagnetic potentials on M , $g_{\alpha\beta}$ as the local components of a Lorentz metric on M , and $\varepsilon = 1$. In this case, the metric \bar{g} given by (1.17) is just the metric used in the classical Kaluza-Klein theory. For this reason, we call (\bar{M}, \bar{g}) , where \bar{g} is given by (1.15) or (1.17), a *general Kaluza-Klein space*.

It is worth mentioning that the above metric \bar{g} can be also considered as an extension of the bulk metric used in the studies of Randall and Sundrum.^{23),24)} The extension is done in two directions. First, we do not exclude the off-diagonal fluctuations of the metric, which are here represented by εA_α . Secondly, we consider on an equal foot, both the compact and non-compact extradimensions. The common characteristic of both theories is that they are based on a non-factorizable background geometry.

Finally, we note that in space-time-matter theories (cf. Wesson²⁸⁾) the metric \bar{g} is represented by a matrix in (1.17), but with some changes in the $(i, 4)$ -entries. These differences are due to the fact that, without loss of generality, we use adapted coordinate systems with respect to which (1.6) is valid.

§2. Horizontal tensor fields on a general Kaluza-Klein space

In this section, we develop a tensor calculus on the horizontal distribution of (\bar{M}, \bar{g}) . A general theory of adapted tensor fields on a foliated manifold was fully developed in the book of Bejancu and Farran.¹⁾

Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space. Denote by $H\bar{M}^*$ the dual vector bundle to $H\bar{M}$. Then a *horizontal tensor field* of type (p, q) on \bar{M} is an $\mathcal{F}(M)$ -multilinear mapping

$$T : \Gamma(H\bar{M}^*)^p \times (\Gamma(H\bar{M}))^q \rightarrow \mathcal{F}(\bar{M}).$$

As a first example of such tensor field, we present the Lorentz metric g on $H\bar{M}$, which is a horizontal tensor field of type $(0, 2)$ on \bar{M} :

$$g : \Gamma(H\bar{M}) \times \Gamma(H\bar{M}) \rightarrow \mathcal{F}(\bar{M}).$$

Moreover, the local components $g_{\alpha\beta}$ of g with respect to the frame field $\{\delta/\delta x^\alpha\}$ on $H\bar{M}$, satisfy some tensorial transformations which are similar to the ones on the spacetime manifold M . Indeed, by using (1.13) and (1.12), we deduce that

$$g_{\alpha\beta} = \tilde{g}_{\gamma\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\mu}{\partial x^\beta}. \tag{2.1}$$

Next, we suppose that X is a horizontal vector field on \bar{M} , that is, we have

$$X = X^\alpha \frac{\delta}{\delta x^\alpha}.$$

Then, by using again (1.12), we deduce that the local components of X satisfy the tensorial transformations

$$\tilde{X}^\gamma = X^\alpha \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}. \tag{2.2}$$

Now, looking at (2.1) and (2.2), and with the usual tensor calculus on M in mind, we can easily imagine the tensorial transformations of any horizontal tensor field on \bar{M} . We have to note that the idea of such horizontal tensor analysis can be found, for the first time, in the paper of Einstein and Bergmann.⁷⁾ The difference is that they consider separately the four-transformations (1.7a) and the cut-transformations (1.7b). Thus our (2.2) is just (24) in 7), and the vector fields given by (1.10) can be found in formula (26) of 7).

Next, by using the local components $g_{\alpha\beta}$ of the Lorentz metric g on $H\bar{M}$, we can construct a horizontal tensor field of type $(2, 0)$, as follows. Let $g^{\alpha\beta}$ be the entries of the inverse of the matrix $[g_{\alpha\beta}]$. Then, by using (2.1), we deduce that

$$g^{\alpha\beta} = \tilde{g}^{\gamma\mu} \frac{\partial x^\alpha}{\partial \tilde{x}^\gamma} \frac{\partial x^\beta}{\partial \tilde{x}^\mu},$$

and, therefore, $g^{\alpha\beta}$ define a horizontal tensor field of type $(2, 0)$. As in the case of usual tensor fields on manifolds, a horizontal tensor field of type $(1, q)$ on \bar{M} can be also thought as an $\mathcal{F}(\bar{M})$ -multilinear mapping

$$T : \Gamma(H\bar{M})^q \rightarrow \Gamma(H\bar{M}).$$

We close this section with a presentation of some horizontal tensor fields which have a great role in the development of our approach. First, by direct calculations using (1.10), we deduce that

$$(a) \left[\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha} \right] = F_{\alpha\beta} \frac{\partial}{\partial x^4}, \quad (b) \left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^4} \right] = B_\alpha \frac{\partial}{\partial x^4}, \tag{2.3}$$

where we put:

$$(a) \quad F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} - A_\alpha \frac{\partial A_\beta}{\partial x^4} + A_\beta \frac{\partial A_\alpha}{\partial x^4} = \frac{\delta A_\beta}{\delta x^\alpha} - \frac{\delta A_\alpha}{\delta x^\beta},$$

$$(b) \quad B_\alpha = \frac{\partial A_\alpha}{\partial x^4}. \tag{2.4}$$

Now, we denote by h and v the projection morphisms of $T\bar{M}$ on $H\bar{M}$ and $V\bar{M}$, respectively. Then, we define

$$F : \Gamma(H\bar{M}) \times \Gamma(H\bar{M}) \rightarrow \Gamma(V\bar{M}); \quad F(hX, hY) = -v[hX, hY], \tag{2.5}$$

for all $X, Y \in \Gamma(T\bar{M})$. By direct calculations using (2.5) and (2.3a), we obtain

$$F \left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right) = F_{\alpha\beta} \frac{\partial}{\partial x^4}. \tag{2.6}$$

Moreover, by using (1.8b) and (1.12) into (2.6), we deduce that $F_{\alpha\beta}$ satisfy

$$F_{\alpha\beta} = \tilde{F}_{\gamma\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\mu}{\partial x^\beta},$$

with respect to the coordinate transformations (1.7). Hence, the functions $F_{\alpha\beta}$, $\alpha, \beta \in \{0, 1, 2, 3\}$, given by (2.4a), define a skew-symmetric horizontal tensor field of type $(0, 2)$ on \bar{M} . If, in particular, A_α are the electromagnetic potentials on M , then from (2.4a) we deduce that $F_{\alpha\beta}$ are the local components of the electromagnetic tensor field on M , that is, we have

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}.$$

For this reason, we call the horizontal tensor field with local components $F_{\alpha\beta}$, given by (2.4a), the *horizontal electromagnetic tensor field* of the general Kaluza-Klein space (\bar{M}, \bar{g}) .

Next, we note that, according to (1.11), the functions A_α do not define a horizontal covector field on (\bar{M}, \bar{g}) . However, taking partial derivatives in (1.11) with respect to x^4 , and using (2.4b) and (1.7), we obtain

$$B_\alpha = \tilde{B}_\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}.$$

Thus, the functions B_α define a horizontal covector field B on \bar{M} , given by

$$B(hX) = B_\alpha X^\alpha, \quad \forall X \in \Gamma(T\bar{M}), \tag{2.7}$$

where X^α are the local components of hX with respect to the frame field $\{\delta/\delta x^\alpha\}$.

Finally, we define the mapping

$$\begin{aligned} D : \Gamma(H\bar{M})^2 \times \Gamma(V\bar{M}) &\rightarrow \mathcal{F}(\bar{M}), \\ D(hX, hY, vZ) &= \frac{1}{2} \{vZ(g(hX, hY)) - g(h[vZ, hX], hY) \\ &\quad - g(h[vZ, hY], hX)\}, \end{aligned} \tag{2.8}$$

for all $X, Y, Z \in \Gamma(T\bar{M})$. It is easy to check that D is $\mathcal{F}(\bar{M})$ -3-linear mapping satisfying

$$D(hX, hY, vZ) = D(hY, hX, vZ). \tag{2.9}$$

Then, by using the Lorentz metric g on $H\bar{M}$ and D given by (2.8), we define another mapping, still denoted by D , and given by

$$\begin{aligned} D : \Gamma(V\bar{M}) \times \Gamma(H\bar{M}) &\rightarrow \Gamma(H\bar{M}), \\ g(D(vX, hY), hZ) &= D(hY, hZ, vX). \end{aligned} \tag{2.10}$$

Locally, we put

$$\begin{aligned} \text{(a)} \quad D \left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}, \frac{\partial}{\partial x^4} \right) &= D_{\alpha\beta}, \\ \text{(b)} \quad D \left(\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\alpha} \right) &= D_{\alpha\gamma} \frac{\delta}{\delta x^\gamma}. \end{aligned} \tag{2.11}$$

Then, by using (2·8), (1·13), (2·3b), (2·10) and (2·11), we obtain

$$(a) D_{\alpha\beta} = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^4}, \quad (b) D_{\alpha}{}^{\gamma} = D_{\alpha\beta} g^{\beta\gamma} = \frac{1}{2} g^{\beta\gamma} \frac{\partial g_{\alpha\beta}}{\partial x^4}. \quad (2.12)$$

Taking into account the $\mathcal{F}(\overline{M})$ -linearity of both D given by (2·8) and (2·10), we deduce that the functions $D_{\alpha\beta}$ and $D_{\alpha}{}^{\gamma}$ given by (2·12) define horizontal tensor fields of type (0, 2) and (1, 1), respectively. From (2·4b) and (2·12a) we deduce the following.

Proposition 2.1 *A general Kaluza-Klein space $(\overline{M}, \overline{g})$ becomes a classical Kaluza-Klein space, if and only if the horizontal tensor fields B and D vanish identically on \overline{M} .*

§3. A remarkable linear connection on $H\overline{M}$

In this section we construct a linear connection on the horizontal distribution $H\overline{M}$, which is going to play, in our approach, the same role as the Levi-Civita connection on the spacetime, in the classical Kaluza-Klein theory. This connection is going to be the main tool in the general Kaluza-Klein theory that we develop in this paper. More precisely, we use it for a study of motions in $(\overline{M}, \overline{g})$ (see §§4, 5 and 6), and for the construction of the horizontal Einstein gravitational tensor field on $H\overline{M}$ which leads us to some horizontal field equations (see §§7, 8, 9 and 10).

Let ∇ be a linear connection on $H\overline{M}$, that is, ∇ is a mapping

$$\nabla : \Gamma(T\overline{M}) \times \Gamma(H\overline{M}) \rightarrow \Gamma(H\overline{M}), \quad (X, hY) \rightarrow \nabla_X hY,$$

satisfying the following conditions:

- (a) ∇ is $\mathcal{F}(\overline{M})$ -linear with respect to the first variable.
- (b) ∇ is a derivation with respect to the second variable, that is, we have

$$\nabla_X(fhY + hZ) = X(f)hY + f\nabla_X hY + \nabla_X hZ,$$

for all $X, Y, Z \in \Gamma(T\overline{M})$.

Then, we define the *torsion tensor field* of ∇ as $\mathcal{F}(\overline{M})$ -bilinear mapping

$$T : \Gamma(T\overline{M}) \times \Gamma(H\overline{M}) \rightarrow \Gamma(H\overline{M}),$$

$$T(X, hY) = \nabla_X hY - \nabla_{hY} hX - h[X, hY], \quad \forall X, Y \in \Gamma(T\overline{M}). \quad (3.1)$$

Also, taking into account that on $H\overline{M}$ there exists a Lorentz metric g , we say that ∇ is a *metric connection* on $H\overline{M}$, if g is parallel with respect to ∇ , that is, we have

$$(\nabla_X g)(hY, hZ) = X(g(hY, hZ)) - g(\nabla_X hY, hZ) - g(hY, \nabla_X hZ) = 0,$$

$$\forall X, Y, Z \in \Gamma(T\overline{M}). \quad (3.2)$$

Now, we can prove the following important result.

Theorem 3.1 *Let $(\overline{M}, \overline{g})$ be a general Kaluza-Klein space endowed with the Lorentz horizontal distribution $(H\overline{M}, g)$. Then there exists a unique linear connection ∇ on $H\overline{M}$ satisfying the following conditions:*

- (i) ∇ is a metric connection.
- (ii) The torsion tensor field of ∇ is given by

$$(a) T(hX, hY) = 0, \quad (b) T(vX, hY) = D(vX, hY), \quad (3.3)$$

for all $X, Y \in \Gamma(T\bar{M})$, where D is the horizontal tensor field defined by (2.10) and (2.8).

Proof. Define the mapping

$$\nabla : \Gamma(T\bar{M}) \times \Gamma(H\bar{M}) \rightarrow \Gamma(H\bar{M}),$$

as follows:

$$\begin{aligned} (a) \quad 2g(\nabla_{hX}hY, hZ) &= hX(g(hY, hZ)) + hY(g(hZ, hX)) \\ &\quad - hZ(g(hX, hY)) + g(h[hX, hY], hZ) \\ &\quad - g(h[hY, hZ], hX) + g(h[hZ, hX], hY), \\ (b) \quad \nabla_{vX}hY &= h[vX, hY] + D(vX, hY), \end{aligned} \quad (3.4)$$

for all $X, Y, Z \in \Gamma(T\bar{M})$. Then it is easy to check that ∇ is a linear connection on $H\bar{M}$ satisfying the conditions (i) and (ii). Now, we suppose that ∇' is another linear connection on $H\bar{M}$ satisfying (i) and (ii). Then, by using (3.3b) and (3.1) for ∇' , we deduce that ∇' satisfies (3.4b). Finally, by using (3.2) and (3.3a) for ∇' , we obtain

$$\begin{aligned} 0 &= (\nabla'_{hX}g)(hY, hZ) + (\nabla'_{hY}g)(hZ, hX) - (\nabla'_{hZ}g)(hX, hY) \\ &= hX(g(hY, hZ)) + hY(g(hZ, hX)) - hZ(g(hX, hY)) \\ &\quad + g(h[hX, hY], hZ) - g(h[hY, hZ], hX) + g(h[hZ, hX], hY) \\ &\quad - 2g(\nabla'_{hX}hY, hZ), \end{aligned}$$

which proves (3.4a) for ∇' . Hence $\nabla' = \nabla$, and the proof is complete. ■

We recall that the *Levi-Civita (Riemannian) connection* $\bar{\nabla}$ on the pseudo-Riemannian manifold (\bar{M}, \bar{g}) is given by (cf. O'Neill,¹⁸ p. 61)

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= X(\bar{g}(Y, Z)) + Y(\bar{g}(Z, X)) - Z(\bar{g}(X, Y)) \\ &\quad + \bar{g}([X, Y], Z) - \bar{g}([Y, Z], X) + \bar{g}([Z, X], Y), \end{aligned} \quad (3.5)$$

for all $X, Y, Z \in \Gamma(T\bar{M})$. Moreover, $\bar{\nabla}$ is the unique linear connection on \bar{M} which is a metric and torsion-free linear connection. Taking into account that ∇ constructed in Theorem 3.1 is a metric connection and satisfies (3.3a), we are entitled to call it the *Riemannian horizontal connection* on (\bar{M}, \bar{g}) . Also, by (3.3) we see that D given by (2.10) and (2.8) can be called the *torsion tensor field* of ∇ .

Now, we prove the following.

Proposition 3.1 *The Levi-Civita connection $\bar{\nabla}$ on (\bar{M}, \bar{g}) and the Riemannian horizontal connection on $(H\bar{M}, g)$ satisfy the identities*

$$\begin{aligned} (a) \quad h\bar{\nabla}_{hX}hY &= \nabla_{hX}hY, \\ (b) \quad \bar{g}(v\bar{\nabla}_{hX}hY, vZ) &= -D(hX, hY, vZ) - \frac{1}{2} \bar{g}(F(hX, hY), vZ), \\ (c) \quad g(h\bar{\nabla}_{vX}hY, hZ) &= g(\nabla_{vX}hY, hZ) + \frac{1}{2} \bar{g}(F(hY, hZ), vX), \end{aligned} \quad (3.6)$$

for all $X, Y, Z \in \Gamma(T\bar{M})$.

Proof. First, we take $X = hX, Y = hY$ and $Z = hZ$ in (3.5) and, by using (1.13), (1.14) and (3.4a), we obtain (3.6a). Then we take $X = hX, Y = hY$ and $Z = vZ$ into (3.5), and, by using (1.13), (1.14), (2.8) and (2.5), we deduce that

$$\begin{aligned} 2\bar{g}(v\bar{\nabla}_{hX}hY, vZ) &= 2\bar{g}(\bar{\nabla}_{hX}hY, vZ) = \{-vZ(g(hX, hY)) + g(h[vZ, hX], hY) \\ &\quad + g(h[vZ, hY], hX)\} + \bar{g}(v[hX, hY], vZ) \\ &= -2D(hX, hY, vZ) - \bar{g}(F(hX, hY), vZ), \end{aligned}$$

which proves (3.6b). Finally, we take $X = vX, Y = hY$ and $Z = hZ$ into (3.5), and, by using (1.13), (1.14), (2.5), (2.8), (2.10) and (3.4b), we obtain

$$\begin{aligned} 2\bar{g}(h\bar{\nabla}_{vX}hY, hZ) &= 2\bar{g}(\bar{\nabla}_{vX}hY, hZ) = vX(g(hY, hZ)) + g(h[vX, hY], hZ) \\ &\quad - g(h[vX, hZ], hY) - \bar{g}(v[hY, hZ], vX) \\ &= 2D(hY, hZ, vX) + 2g(h[vX, hY], hZ) + \bar{g}(F(hY, hZ), vX) \\ &= 2g(\nabla_{vX}hY, hZ) + \bar{g}(F(hY, hZ), vX), \end{aligned}$$

which proves (3.6c). ■

Taking into account that ∇ and $\bar{\nabla}$ are used in a study of both the equations of motion and the Einstein equations on (\bar{M}, \bar{g}) , we close this section with a presentation of these objects by using local coordinates on \bar{M} . Let (x^α, x^4) be an adapted coordinate system in (\bar{M}, \bar{g}) and $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ the corresponding adapted frame field on (\bar{M}, \bar{g}) . Then we put

$$(a) \quad \nabla_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = \Gamma_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}, \quad (b) \quad \nabla_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} = \Gamma_\alpha{}^\gamma{}_4 \frac{\delta}{\delta x^\gamma}. \quad (3.7)$$

Next, we take $hX = \delta/\delta x^\beta, hY = \delta/\delta x^\alpha, hZ = \delta/\delta x^\mu$ in (3.4a) and by using (3.7a), (1.13), (1.14) and (2.3a), we obtain

$$\Gamma_\alpha{}^\gamma{}_\beta = \frac{1}{2} g^{\gamma\mu} \left\{ \frac{\delta g_{\mu\alpha}}{\delta x^\beta} + \frac{\delta g_{\mu\beta}}{\delta x^\alpha} - \frac{\delta g_{\alpha\beta}}{\delta x^\mu} \right\}. \quad (3.8)$$

Similarly, we take $vX = \partial/\partial x^4$ and $hY = \delta/\delta x^\alpha$ in (3.4b), and by using (3.7b), (2.3b) and (2.11b), we deduce that

$$\Gamma_\alpha{}^\gamma{}_4 = D_\alpha{}^\gamma. \quad (3.9)$$

Now, we take $hY = \delta/\delta x^\beta, hZ = \delta/\delta x^\alpha$, and in turn $X = \delta/\delta x^\nu$ and $X = \partial/\partial x^4$ into (3.2), and by using (1.13), (3.7) and (3.9), we obtain

$$\begin{aligned} (a) \quad g_{\alpha\beta|\nu} &= \frac{\delta g_{\alpha\beta}}{\delta x^\nu} - g_{\gamma\beta} \Gamma_\alpha{}^\gamma{}_\nu - g_{\alpha\gamma} \Gamma_\beta{}^\gamma{}_\nu = 0, \\ (b) \quad g_{\alpha\beta|4} &= \frac{\partial g_{\alpha\beta}}{\partial x^4} - g_{\gamma\beta} D_\alpha{}^\gamma - g_{\alpha\gamma} D_\beta{}^\gamma = 0. \end{aligned} \quad (3.10)$$

Remark 3.1 It is important to note that the local coefficients (3·8) have been defined for the first time by Einstein and Bergmann⁷⁾ in formula (27b). However, the covariant differentiation defined in 7) is only *h*-metric, that is, (3·10a) is satisfied while (3·10b), in general, is not satisfied. This is because in 7) the covariant derivative of horizontal tensor fields with respect to x^4 is just the partial derivative. ■

As a consequence of (3·10), we have

$$(a) \quad g^{\alpha\beta}{}_{|\gamma} = 0, \quad (b) \quad g^{\alpha\beta}{}_{|4} = 0. \quad (3.11)$$

Now, we note that for raising and lowering Greek indices, we use $g^{\alpha\beta}$ and $g_{\alpha\beta}$, respectively, as in the following examples:

$$\begin{aligned} (a) \quad F_\alpha{}^\gamma &= F_{\alpha\beta}g^{\beta\gamma}, & (b) \quad F^{\mu\gamma} &= g^{\mu\alpha}F_\alpha{}^\gamma, \\ (c) \quad B^\gamma &= g^{\gamma\alpha}B_\alpha, & (d) \quad D_\alpha{}^\gamma &= D_{\alpha\beta}g^{\beta\gamma}. \end{aligned} \quad (3.12)$$

Next, we state the following.

Theorem 3.2 *The Levi-Civita connection on a general Kaluza-Klein space (\bar{M}, \bar{g}) is given by*

$$\begin{aligned} (a) \quad \bar{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} &= \Gamma_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) \frac{\partial}{\partial x^4}, \\ (b) \quad \bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} &= \left(D_\alpha{}^\gamma + \frac{\varepsilon}{2} F_\alpha{}^\gamma \right) \frac{\delta}{\delta x^\gamma} - B_\alpha \frac{\partial}{\partial x^4}, \\ (c) \quad \bar{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^4} &= \left(D_\alpha{}^\gamma + \frac{\varepsilon}{2} F_\alpha{}^\gamma \right) \frac{\delta}{\delta x^\gamma}, \\ (d) \quad \bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4} &= \varepsilon B^\gamma \frac{\delta}{\delta x^\gamma}. \end{aligned} \quad (3.13)$$

Proof. First, we put

$$\bar{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = \bar{\Gamma}_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + \bar{\Gamma}_\alpha{}^4{}_\beta \frac{\partial}{\partial x^4}. \quad (3.14)$$

Then, by (3·6a) and (3·7a), we obtain

$$\bar{\Gamma}_\alpha{}^\gamma{}_\beta = \Gamma_\alpha{}^\gamma{}_\beta. \quad (3.15)$$

Now, by using (3·6b), (2·11a), (2·6) and (1·6), we infer that

$$\bar{g} \left(v \bar{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4} \right) = \frac{\varepsilon}{2} F_{\alpha\beta} - D_{\alpha\beta}. \quad (3.16)$$

Taking into account (3·14) into (3·16), and by using (1·6), we obtain

$$\bar{\Gamma}_\alpha{}^4{}_\beta = \frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta}. \quad (3.17)$$

Thus (3·13a) is deduced from (3·14), by using (3·15) and (3·17). Next, we put

$$\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} = \bar{\Gamma}_\alpha{}^\gamma{}_4 \frac{\delta}{\delta x^\gamma} + \bar{\Gamma}_\alpha{}^4{}_4 \frac{\partial}{\partial x^4}. \quad (3.18)$$

Then, we take $vX = \partial/\partial x^4$, $hY = \delta/\delta x^\alpha$ and $hZ = \delta/\delta x^\mu$ in (3.6c) and by using (3.18), (3.7b), (1.13), (3.9), (2.6) and (1.6), we deduce that

$$\bar{\Gamma}_\alpha{}^\nu{}_4 g_{\nu\mu} = D_\alpha{}^\nu g_{\nu\mu} + \frac{\varepsilon}{2} F_{\alpha\mu}.$$

Contracting this by $g^{\mu\gamma}$ and taking into account (3.12a), we obtain

$$\bar{\Gamma}_\alpha{}^\gamma{}_4 = D_\alpha{}^\gamma + \frac{\varepsilon}{2} F_\alpha{}^\gamma. \tag{3.19}$$

Now, by using (3.18), (1.6), (3.5) and (2.3b), we infer that

$$\bar{\Gamma}_\alpha{}^4{}_4 = \varepsilon \bar{g} \left(\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4} \right) = -\varepsilon \bar{g} \left(\left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4} \right], \frac{\partial}{\partial x^4} \right) = -B_\alpha. \tag{3.20}$$

Thus (3.13b) is obtained from (3.18) by using (3.19) and (3.20). Taking into account that $\bar{\nabla}$ is torsion-free, and by using (3.13b) and (2.3b), we deduce that

$$\bar{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^4} = \bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} + \left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4} \right] = \left(D_\alpha{}^\gamma + \frac{\varepsilon}{2} F_\alpha{}^\gamma \right) \frac{\delta}{\delta x^\gamma},$$

which proves (3.13c). Next, due to (1.6), we have

$$\bar{g} \left(\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4} \right) = 0. \tag{3.21}$$

Hence we have

$$\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4} = \bar{\Gamma}_4{}^\gamma{}_4 \frac{\delta}{\delta x^\gamma}, \tag{3.22}$$

which via (1.13) implies

$$\bar{g} \left(\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\alpha} \right) = g_{\alpha\gamma} \bar{\Gamma}_4{}^\gamma{}_4. \tag{3.23}$$

On the other hand, taking into account that \bar{g} is parallel with respect to $\bar{\nabla}$, and using (1.14), (3.13b) and (1.6), we obtain

$$\bar{g} \left(\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\alpha} \right) = -\bar{g} \left(\frac{\partial}{\partial x^4}, \bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} \right) = \varepsilon B_\alpha. \tag{3.24}$$

Comparing (3.23) with (3.24) and using (3.12c), we deduce that

$$\bar{\Gamma}_4{}^\gamma{}_4 = \varepsilon B^\gamma. \tag{3.25}$$

Finally, by using (3.25) into (3.22), we obtain (3.13d). ■

Now, we note that covariant derivatives of horizontal tensor fields are horizontal tensor fields too. More precisely, the covariant derivatives $F_{\alpha\beta|\gamma}$, $D_{\alpha\beta|\gamma}$ and $B_{\alpha|\gamma}$

define horizontal tensor fields of types (0, 3), (0, 3) and (0, 2), respectively, and they are given by

$$\begin{aligned}
 \text{(a)} \quad F_{\alpha\beta|\gamma} &= \frac{\delta F_{\alpha\beta}}{\delta x^\gamma} - F_{\mu\beta} \Gamma_{\alpha}{}^\mu{}_\gamma - F_{\alpha\mu} \Gamma_{\beta}{}^\mu{}_\gamma, \\
 \text{(b)} \quad D_{\alpha\beta|\gamma} &= \frac{\delta D_{\alpha\beta}}{\delta x^\gamma} - D_{\mu\beta} \Gamma_{\alpha}{}^\mu{}_\gamma - D_{\alpha\mu} \Gamma_{\beta}{}^\mu{}_\gamma, \\
 \text{(c)} \quad B_{\alpha|\gamma} &= \frac{\delta B_\alpha}{\delta x^\gamma} - B_\mu \Gamma_{\alpha}{}^\mu{}_\gamma.
 \end{aligned} \tag{3.26}$$

Similarly, $F_{\alpha\beta|4}$, $D_{\alpha\beta|4}$ and $B_{\alpha|4}$ define horizontal tensor fields of type (0, 2), (0, 2) and (0, 1), respectively, and they are given by

$$\begin{aligned}
 \text{(a)} \quad F_{\alpha\beta|4} &= \frac{\partial F_{\alpha\beta}}{\partial x^4} - F_{\mu\beta} D_{\alpha}{}^\mu - F_{\alpha\mu} D_{\beta}{}^\mu, \\
 \text{(b)} \quad D_{\alpha\beta|4} &= \frac{\partial D_{\alpha\beta}}{\partial x^4} - D_{\mu\beta} D_{\alpha}{}^\mu - D_{\alpha\mu} D_{\beta}{}^\mu = \frac{\partial D_{\alpha\beta}}{\partial x^4} - 2D_{\alpha\mu} D_{\beta}{}^\mu, \\
 \text{(c)} \quad B_{\alpha|4} &= \frac{\partial B_\alpha}{\partial x^4} - B_\mu D_{\alpha}{}^\mu.
 \end{aligned} \tag{3.27}$$

The covariant derivatives (3.26) and (3.27) will have a great role in §§8, 9 and 10.

§4. Equations of motion in a general Kaluza-Klein space

In the previous sections, we present our new point of view on the geometry of a general Kaluza-Klein space (\bar{M}, \bar{g}) . In summary, this is based on the following:

- a. The splitting of $T\bar{M}$ into horizontal and vertical distributions.
- b. The existence of adapted coordinate systems and adapted frame fields.
- c. The new geometric objects: Riemannian horizontal connection ∇ and the horizontal tensor fields F , B and D .

The present section is the first one in which we show the powerful of the new method that we introduce for studying general Kaluza-Klein theories. By this method, we write down, in covariant form, the equations of motion in (\bar{M}, \bar{g}) , analyzing two classes of geodesics. It is interesting to note that the geodesics of (\bar{M}, \bar{g}) which are tangent to the horizontal distribution $H\bar{M}$, must be autoparallels with respect to the Riemannian horizontal connection ∇ on $H\bar{M}$. The projection of the geodesics of (\bar{M}, \bar{g}) on the base manifold M gives two classes of motions on M , which have physical interpretations for some particular cases.

Let \bar{C} be a smooth curve in \bar{M} given by the equations:

$$x^i = x^i(t), \quad i \in \{0, 1, 2, 3, 4\}. \tag{4.1}$$

Then, the tangent vector field d/dt to \bar{C} is expressed with respect to the natural frame field $\{\partial/\partial x^\alpha, \partial/\partial x^4\}$ as follows:

$$\frac{d}{dt} = \frac{dx^\alpha}{dt} \frac{\partial}{\partial x^\alpha} + \frac{dx^4}{dt} \frac{\partial}{\partial x^4}. \tag{4.2}$$

The expression (4.2) of d/dt was used in the whole literature on the space-time-matter theory, in order to write down the equations of motion in (\bar{M}, \bar{g}) (cf. Wesson²⁸).

Moreover, most of the results have been stated by using the arc-length parameter on \overline{C} . The lack of a tensorial calculus on $(\overline{M}, \overline{g})$ with respect to the coordinate transformations (1.7), raised the problem of covariance of these equations. By our method we overtake this deficiency, and present here the equations of motion on $(\overline{M}, \overline{g})$ in a form that it is invariant with respect to (1.7).

First, by using (1.10) into (4.2), we obtain

$$\frac{d}{dt} = \frac{dx^\alpha}{dt} \frac{\delta}{\delta x^\alpha} + \frac{\delta x^4}{\delta t} \frac{\partial}{\partial x^4}, \quad (4.3)$$

where we put

$$\frac{\delta x^4}{\delta t} = \frac{dx^4}{dt} + A_\alpha \frac{dx^\alpha}{dt}. \quad (4.4)$$

Then we prove the following.

Proposition 4.1 *Let \overline{C} be a curve in a general Kaluza-Klein space $(\overline{M}, \overline{g})$. Then the covariant derivatives of the adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ with respect to the Levi-Civita connection $\overline{\nabla}$ on $(\overline{M}, \overline{g})$ in the direction of the tangent vector field to \overline{C} are given by*

$$\begin{aligned} \text{(a)} \quad \overline{\nabla}_{\frac{d}{dt}} \frac{\delta}{\delta x^\alpha} &= \left\{ \Gamma_{\alpha\gamma\beta} \frac{dx^\beta}{dt} + \left(D_{\alpha\gamma} + \frac{\varepsilon}{2} F_{\alpha\gamma} \right) \frac{\delta x^4}{\delta t} \right\} \frac{\delta}{\delta x^\gamma} \\ &\quad + \left\{ \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) \frac{dx^\beta}{dt} - B_\alpha \frac{\delta x^4}{\delta t} \right\} \frac{\partial}{\partial x^4}, \\ \text{(b)} \quad \overline{\nabla}_{\frac{d}{dt}} \frac{\partial}{\partial x^4} &= \left\{ \left(D_{\alpha\gamma} + \frac{\varepsilon}{2} F_{\alpha\gamma} \right) \frac{dx^\alpha}{dt} + \varepsilon B^\gamma \frac{\delta x^4}{\delta t} \right\} \frac{\delta}{\delta x^\gamma}. \end{aligned} \quad (4.5)$$

Proof. Both formulae in (4.5) are obtained by direct calculations. First, by using (4.3), (3.13a) and (3.13b), we obtain (4.5a). Then, (4.5b) is deduced in a similar way, by using (4.3), (3.13c) and (3.13d). ■

Now, we can state the main result of this section.

Theorem 4.1 *The equations of motion in a general Kaluza-Klein space $(\overline{M}, \overline{g})$ are expressed as follows:*

$$\begin{aligned} \text{(a)} \quad \frac{d^2 x^\gamma}{dt^2} + \Gamma_{\alpha\gamma\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + (2D_{\alpha\gamma} + \varepsilon F_{\alpha\gamma}) \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} \\ + \varepsilon B^\gamma \left(\frac{\delta x^4}{\delta t} \right)^2 = 0, \\ \text{(b)} \quad \frac{d}{dt} \left(\frac{\delta x^4}{\delta t} \right) - \varepsilon D_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} - B_\alpha \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} = 0, \end{aligned} \quad (4.6)$$

where $(\Gamma_{\alpha\gamma\beta}, D_{\alpha\gamma})$ are the local coefficients of the Riemannian horizontal connection ∇ on $H\overline{M}$, and $F_{\alpha\gamma}, B_\alpha, D_{\alpha\beta}$ are the local components of the horizontal tensor fields defined in §2.

Proof. As it is well known, the equations of geodesics of pseudo-Riemannian manifold (\bar{M}, \bar{g}) are given by (cf. O'Neill,¹⁸) p. 67)

$$\bar{\nabla}_{\frac{d}{dt}} \frac{d}{dt} = 0, \tag{4.7}$$

where $\bar{\nabla}$ is the Levi-Civita connection on (\bar{M}, \bar{g}) . By using (4.3) and (4.5), and taking into account that $\bar{\nabla}$ is a linear connection and that $F_{\alpha\beta}$ are skew-symmetric, we deduce that:

$$\begin{aligned} \bar{\nabla}_{\frac{d}{dt}} \frac{d}{dt} &= \bar{\nabla}_{\frac{d}{dt}} \left(\frac{dx^\alpha}{dt} \frac{\delta}{\delta x^\alpha} + \frac{\delta x^4}{\delta t} \frac{\partial}{\partial x^4} \right) = \frac{d^2 x^\gamma}{dt^2} \frac{\delta}{\delta x^\gamma} \\ &\quad + \frac{d}{dt} \left(\frac{\delta x^4}{\delta t} \right) \frac{\partial}{\partial x^4} + \frac{dx^\alpha}{dt} \bar{\nabla}_{\frac{d}{dt}} \frac{\delta}{\delta x^\alpha} + \frac{\delta x^4}{\delta t} \bar{\nabla}_{\frac{d}{dt}} \frac{\partial}{\partial x^4} \\ &= \left\{ \frac{d^2 x^\gamma}{dt^2} + \Gamma_{\alpha\gamma\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + (2D_{\alpha\gamma} + \varepsilon F_{\alpha\gamma}) \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} \right. \\ &\quad \left. + \varepsilon B^\gamma \left(\frac{\delta x^4}{\delta t} \right)^2 \right\} \frac{\delta}{\delta x^\gamma} \\ &\quad + \left\{ \frac{d}{dt} \left(\frac{\delta x^4}{\delta t} \right) - \varepsilon D_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} - B_\alpha \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} \right\} \frac{\partial}{\partial x^4}. \end{aligned} \tag{4.8}$$

Thus, (4.6) is obtained from (4.7) by using (4.8), and taking into account that $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ is a local frame field on \bar{M} . ■

Before we study the above equations of motion, we compare them with what is known in literature. The main difference between (4.6) and some other equations of motion in (\bar{M}, \bar{g}) is that the coefficients $\Gamma_{\alpha\gamma\beta}$ from (4.6a) define indeed a linear connection ∇ on $H\bar{M}$ which we call the Riemannian horizontal connection on (\bar{M}, \bar{g}) . By using (3.7a) and (1.12), we deduce that $\Gamma_{\alpha\gamma\beta}$ satisfy the following transformations with respect to (1.7):

$$\Gamma_{\alpha\gamma\beta} \frac{\partial \tilde{x}^\mu}{\partial x^\gamma} = \tilde{\Gamma}_{\nu\lambda} \frac{\partial \tilde{x}^\nu}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\beta} + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta}. \tag{4.9}$$

In all the space-time-matter theories there are used such local coefficients, which we denote here by $\Gamma^*_{\alpha\gamma\beta}$ and they are given by similar formulas as the Christoffel symbols in the 4-dimensional spacetime, that is, we have

$$\Gamma^*_{\alpha\gamma\beta} = \frac{1}{2} g^{\gamma\mu} \left\{ \frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right\}. \tag{4.10}$$

Unfortunately, the functions $\Gamma^*_{\alpha\gamma\beta}$ given by (4.10) do not satisfy the transformations (4.9), unless $g_{\alpha\beta}$ are independent of the fifth coordinate.

Another difference is that the motions in the spacetime M will be obtained here by the projection of geodesics of (\bar{M}, \bar{g}) on M . So far, these motions were considered

to be induced on M by taking $x^4 = k$, where k is a constant, that is M is thought as a submanifold of (\bar{M}, \bar{g}) . However, according to (1.17), $g_{\alpha\beta}(x^\gamma, k)$ do not represent the local components of the induced Lorentz metric on M by \bar{g} , unless $A_\alpha = 0$ on M for all $\alpha \in \{0, 1, 2, 3\}$.

Also, in this approach we study the role of $A_\alpha(x)$ into the general Kaluza-Klein theory. In most of the other theories, these objects vanish on \bar{M} . Many papers have been concerned with the so called *canonical metric* (Mashhoon and Wesson,¹⁷) Ponce de Leon,²¹) Liko, Overduin, and Wesson,¹⁴) and Chapter 6 in Wesson²⁸) given by

$$\bar{g} = \frac{(x^4)^2}{L^2} g_{\alpha\beta}(x^\alpha, x^4) dx^\alpha dx^\beta - (dx^4)^2, \tag{4.11}$$

where L is a non-zero constant introduced for dimensional consistency. Finally, in the alternative to compactification presented by Randall and Sundrum,^{23),24}) the pseudo-Riemannian metric proposed in the 5-dimensional space is also considered with $A_\alpha = 0$, $\alpha \in \{0, 1, 2, 3\}$.

Now, we come back to the study of equations of motion (4.6), taking into account the role of the horizontal distribution into the classification of geodesics of (\bar{M}, \bar{g}) . First, we say that a curve \bar{C} on (\bar{M}, \bar{g}) is a *horizontal curve*, if it is tangent to the horizontal distribution, at any of its points. Thus, by (4.3) and (4.4), we deduce that \bar{C} is a horizontal curve, if and only if at any point of \bar{C} , one of the following conditions is satisfied:

$$\frac{d}{dt} = \frac{dx^\alpha}{dt} \frac{\delta}{\delta x^\alpha}, \tag{4.12}$$

or

$$\frac{\delta x^4}{\delta t} = \frac{dx^4}{dt} + A_\alpha \frac{dx^\alpha}{dt} = 0. \tag{4.13}$$

A geodesic \bar{C} of (\bar{M}, \bar{g}) , which is also a horizontal curve, is called a *horizontal geodesic*.

Next, by using (4.13) into (4.6), we obtain the following interesting corollary.

Corollary 4.1 *Let \bar{C} be a horizontal curve given by (4.1) in a general Kaluza-Klein space (\bar{M}, \bar{g}) . Then \bar{C} is a horizontal geodesic of (\bar{M}, \bar{g}) , if and only if it satisfies (4.13) and the following equations:*

$$\begin{aligned} \text{(a)} \quad & \frac{d^2 x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0, \\ \text{(b)} \quad & D_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0. \end{aligned} \tag{4.14}$$

It is noteworthy that the equations in (4.14a) are related to the Riemannian horizontal connection ∇ on $H\bar{M}$. To show this, we give the following definition. We say that a horizontal curve \bar{C} in (\bar{M}, \bar{g}) is an *autoparallel curve* with respect to ∇ , if we have

$$\nabla_{\frac{d}{dt}} \frac{d}{dt} = 0, \tag{4.15}$$

where d/dt is given by (4.12). Then, by direct calculations using (4.12) and (3.7a), we deduce that (4.14a) and (4.15) are equivalent. Hence, we can state the following.

Corollary 4.2 *Let \bar{C} be a horizontal curve in (\bar{M}, \bar{g}) . Then \bar{C} is a horizontal geodesic of (\bar{M}, \bar{g}) , if and only if it is an autoparallel curve with respect to the Riemannian horizontal connection ∇ on $H\bar{M}$ and satisfies (4.14b).*

An interesting particular case deserves to be considered into the study. Namely, we suppose that the horizontal electromagnetic tensor field F vanishes identically on \bar{M} . Thus, according to (2.6) we have

$$F_{\alpha\beta} = 0, \quad \forall \alpha, \beta \in \{0, 1, 2, 3\}. \tag{4.16}$$

In this case, by (2.5) we conclude that the horizontal distribution is integrable. Then, a horizontal geodesic \bar{C} of (\bar{M}, \bar{g}) must live entirely in a leaf of $H\bar{M}$. Indeed, in this case, two points from different leaves cannot be joint by a horizontal curve.

Thus, we can state the following.

Proposition 4.2 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space, whose horizontal electromagnetic tensor field F vanishes identically on \bar{M} . Then the horizontal distribution is integrable, and any horizontal geodesic \bar{C} must live in a leaf of $H\bar{M}$.*

The second category of geodesics of (\bar{M}, \bar{g}) contains all the geodesics that are not tangent to $H\bar{M}$ on their entire length. We say that \bar{C} is a *non-horizontal curve* if there exists a point $\bar{x} \in \bar{C}$ such that d/dt given by (4.3) is not tangent to $H\bar{M}$ at \bar{x} . Thus, \bar{C} is a non-horizontal curve, if and only if we have

$$\frac{\delta x^4}{\delta t} = \frac{dx^4}{dt} + A_\alpha \frac{dx^\alpha}{dt} \neq 0, \quad \forall t \in I, \tag{4.17}$$

where I is a subinterval of $[a, b]$. A geodesic of (\bar{M}, \bar{g}) , which is a non-horizontal curve, is called a *non-horizontal geodesic*. Then, from Theorem 4.1 we deduce the following corollary.

Corollary 4.3 *A curve \bar{C} is a non-horizontal geodesic of a Kaluza-Klein space (\bar{M}, \bar{g}) , if and only if it is a solution of the system of differential equations (4.6), subject to the constraint (4.17) at least on a neighborhood of one of its points.*

Next, we consider a curve \bar{C} in \bar{M} given by Eq. (4.1), and suppose that the vector field

$$\frac{d^*}{dt} = \frac{dx^\alpha}{dt} \frac{\delta}{\delta x^\alpha}, \tag{4.18}$$

is non-zero at any point of \bar{C} . Then, we call \bar{C} a *projectable curve* on the base manifold M . In this case, there exists a curve C in M which is obtained by the projection of \bar{C} on M , that is, its equations are given by

$$x^\alpha = x^\alpha(t), \quad t \in [a, b], \quad \alpha \in \{0, 1, 2, 3\}. \tag{4.19}$$

We say that C determines an *induced motion* in M . Clearly, any horizontal geodesic is a projectable curve. Also, any non-horizontal geodesic is projectable, if and only if it is not tangent to $V\bar{M}$ at any of its points.

§5. The Einstein-Bergmann generalization of Kaluza-Klein theory

In 1938, Einstein and Bergmann⁷⁾ studied a generalization of the Kaluza-Klein theory. According to the terminology of the present paper, the authors supposed that the following conditions are satisfied:

- (i) The transformations of coordinates in \overline{M} are the four- and cut-transformations given by (1.7).
- (ii) The local components $g_{\alpha\beta}$ of the Lorentz metric on $H\overline{M}$ are periodic functions of x^4 .
- (iii) The functions A_α from (1.17) do not depend on x^4 .

From now on, a general Kaluza-Klein space $(\overline{M}, \overline{g})$ satisfying the conditions (i), (ii) and (iii) will be called an *Einstein-Bergmann space*. From condition (iii) and (2.4b), we deduce that in this section we have

$$B_\alpha = 0, \quad \forall \alpha \in \{0, 1, 2, 3\}. \tag{5.1}$$

Remark 5.1 The condition (5.1) is invariant with respect to four- and cut-transformations (1.7). This is a direct consequence of the fact that B_α are the local components of a horizontal covector field on \overline{M} . ■

By using (5.1) and (2.4a), we deduce that the horizontal electromagnetic tensor field F has the local components

$$F_{\alpha\beta}(x^\mu) = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}. \tag{5.2}$$

However, $A_\alpha(x^\mu)$ are not the local components of a horizontal covector field, since they still satisfy (1.11) with respect to the transformations (1.7). Now, by using (5.1) into (4.6), we deduce the following.

Theorem 5.1 *The equations of motion in an Einstein-Bergmann space $(\overline{M}, \overline{g})$ are given by*

$$\begin{aligned} \text{(a)} \quad & \frac{d^2 x^\gamma}{dt^2} + \Gamma_\alpha^\gamma{}_\beta(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + (2D_\alpha^\gamma(x^i) + \varepsilon F_\alpha^\gamma(x^i)) \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} = 0, \\ \text{(b)} \quad & \frac{d}{dt} \left(\frac{\delta x^4}{\delta t} \right) = \varepsilon D_{\alpha\beta}(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}. \end{aligned} \tag{5.3}$$

Equation (5.3) does not appear in the paper of Einstein-Bergmann.⁷⁾ However, the coefficients $\Gamma_\alpha^\gamma{}_\beta(x^i)$ of the Riemannian horizontal connection ∇ on $H\overline{M}$ coincide with the coefficients given by formula (27b) in 7). In (5.3), there are also involved $D_{\alpha\beta}(x^i)$ and $D_\alpha^\gamma(x^i)$ given by (2.12). Finally, though $F_{\alpha\beta}$ from (5.2) do not depend on x^4 , the functions F_α^γ are given by

$$F_\alpha^\gamma(x^i) = g^{\gamma\beta}(x^i) F_{\alpha\beta}(x^\mu),$$

and, therefore, they are not functions of (x^0, x^1, x^2, x^3) alone.

Next, we analyze the two categories of geodesics on $(\overline{M}, \overline{g})$. First, taking into account that horizontal geodesics satisfy (4.13), from Theorem 5.1 we obtain the following corollary.

Corollary 5.1 *Let (\bar{M}, \bar{g}) be an Einstein-Bergmann space. Then \bar{C} is a horizontal geodesic, if and only if it satisfies the system of differential equations:*

$$\begin{aligned} \text{(a)} \quad & \frac{dx^4}{dt} + A_\alpha(x^\mu) \frac{dx^\alpha}{dt} = 0, \\ \text{(b)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\beta}^\gamma(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0, \\ \text{(c)} \quad & D_{\alpha\beta}(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0. \end{aligned} \tag{5.4}$$

Remark 5.2 According to the discussion from the previous section, from (5.4b) we deduce that the horizontal geodesics of an Einstein-Bergmann space are necessarily autoparallel of the Riemannian horizontal connection ∇ on $H\bar{M}$. ■

Next, we consider the non-horizontal geodesics of (\bar{M}, \bar{g}) satisfying the constraint

$$\frac{dx^4}{dt} + A_\alpha(x^\mu) \frac{dx^\alpha}{dt} = c, \tag{5.5}$$

where c is a non-zero constant. Then, from Theorem 5.1 we deduce the following corollary.

Corollary 5.2 *Any non-horizontal geodesic of the Einstein-Bergmann space (\bar{M}, \bar{g}) , subject to the constraint (5.5), is a solution of the system formed by (5.5) and the equations:*

$$\begin{aligned} \text{(a)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\beta}^\gamma(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + c(2D_\alpha^\gamma(x^i) + \varepsilon F_\alpha^\gamma(x^i)) \frac{dx^\alpha}{dt} = 0, \\ \text{(b)} \quad & D_{\alpha\beta}(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0. \end{aligned} \tag{5.6}$$

In particular, we suppose that $g_{\alpha\beta}$ from (1.13) are functions of (x^0, x^1, x^2, x^3) alone, that is, by (2.12) we have

$$D_{\alpha\beta} = 0, \quad \text{for all } \alpha, \beta \in \{0, 1, 2, 3\}. \tag{5.7}$$

In this case, $g_{\alpha\beta}(x^\mu)$ define a Lorentz metric g on M given by

$$g \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = g_{\alpha\beta}(x^\mu). \tag{5.8}$$

Also, we suppose that $A_\alpha(x^\mu)$ are the local components of a covector field on M , and hence they are thought as *electromagnetic potentials* on the spacetime (M, g) . In this particular case, the Einstein-Bergmann space (\bar{M}, \bar{g}) becomes the *classical Kaluza-Klein space*. Thus, taking into account (5.7), from Theorem 5.1 we deduce the following.

Theorem 5.2 *The equations of motion in a classical Kaluza-Klein space (\bar{M}, \bar{g}) are given by*

$$\begin{aligned} \text{(a)} \quad & \frac{dx^4}{dt} + A_\alpha(x^\mu) \frac{dx^\alpha}{dt} = c, \\ \text{(b)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\beta}^\gamma(x^\mu) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + c\varepsilon F_\alpha^\gamma(x^\mu) \frac{dx^\alpha}{dt} = 0, \end{aligned} \tag{5.9}$$

where c is any real constant.

From this theorem we obtain the following characterizations of horizontal and non-horizontal geodesics of a classical Kaluza-Klein space.

Corollary 5.3 *Let (\bar{M}, \bar{g}) be the classical Kaluza-Klein space. Then we have the following assertions:*

(a) *The horizontal geodesics of (\bar{M}, \bar{g}) are given by the system*

$$\begin{aligned} \text{(a)} \quad & \frac{dx^4}{dt} + A_\alpha(x^\mu) \frac{dx^\alpha}{dt} = 0, \\ \text{(b)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta(x^\mu) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0. \end{aligned} \tag{5.10}$$

(b) *The non-horizontal geodesics of (\bar{M}, \bar{g}) are given by the system (5.9) with $c \neq 0$.*

From the assertion(a) we deduce the following.

Corollary 5.4 *Any motion of the Lorentz manifold (M, g) is the projection of a horizontal geodesic of the classical Kaluza-Klein space (\bar{M}, \bar{g}) , and vice versa.*

Finally, let q and m be the charge and mass of the particle in the spacetime (M, g) . Then the Lorentz force equations of (M, g) with electromagnetic potentials $A_\alpha(x^\mu)$ are expressed as follows:

$$\frac{d^2x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta(x^\mu) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = \frac{q}{m} F_\alpha{}^\gamma(x^\mu) \frac{dx^\alpha}{dt}. \tag{5.11}$$

Thus, from assertion (b) of Corollary 5.3 we deduce the following.

Corollary 5.5 *The solutions of the Lorentz force equations (5.11) on the spacetime (M, g) coincide with the projection of the non-horizontal geodesics of (\bar{M}, \bar{g}) subject to the constraint*

$$\frac{dx^4}{dt} + A_\alpha(x^\mu) \frac{dx^\alpha}{dt} = -\varepsilon \frac{q}{m}. \tag{5.12}$$

Now, we come back to the Einstein-Bergmann space (\bar{M}, \bar{g}) and consider the constraint (5.12). Then, from Corollary 5.2 we deduce the following.

Corollary 5.6 *Any non-horizontal geodesic of the Einstein-Bergmann space (\bar{M}, \bar{g}) , subject to the constraint (5.12), is a solution of the system formed by (5.12) and the equations:*

$$\begin{aligned} \text{(a)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = \frac{q}{m} (2\varepsilon D_\alpha{}^\gamma(x^i) + F_\alpha{}^\gamma(x^i)) \frac{dx^\alpha}{dt}, \\ \text{(b)} \quad & D_{\alpha\beta}(x^i) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0. \end{aligned} \tag{5.13}$$

Comparing (5.11) with (5.13), we are entitled to call (5.13) the Lorentz force equations in an Einstein-Bergmann space (\bar{M}, \bar{g}) . The projections of the solutions of (5.12) and (5.13) on the base manifold M are generalizations of the solutions of the usual Lorentz force equations on a spacetime.

§6. General Kaluza-Klein spaces with bundle-like metrics

Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space whose horizontal tensor field D given by (2·8) vanishes identically on \bar{M} . Then, by (2·12) we deduce that

$$(a) D_{\alpha\beta} = 0, \quad \text{or equivalently,} \quad (b) \frac{\partial g_{\alpha\beta}}{\partial x^4} = 0, \quad \forall \alpha, \beta \in \{0, 1, 2, 3\}. \quad (6.1)$$

According to a well known result on foliated manifolds (cf. Bejancu and Farran,¹ p. 111), the pseudo-Riemannian metric \bar{g} satisfying (6·1b) is *bundle-like* for the foliation determined by the vertical distribution $V\bar{M}$ over \bar{M} . In this case, the geodesics of (\bar{M}, \bar{g}) have an interesting geometric property which we state in the next theorem.

Theorem 6.1 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Suppose that a geodesic \bar{C} of (\bar{M}, \bar{g}) is tangent to the horizontal distribution $H\bar{M}$ at one of its points. Then \bar{C} is a horizontal geodesic of (\bar{M}, \bar{g}) .*

Proof. According to a result of Reinhart,²⁵ each geodesic of (\bar{M}, \bar{g}) which is orthogonal to a leaf of the foliation determined by $V\bar{M}$, remains orthogonal to any leaf of the foliation for its entire length. As $H\bar{M}$ and $V\bar{M}$ are orthogonal with respect to \bar{g} , the assertion of the theorem is a direct consequence of this result. ■

From the above theorem, we conclude that: *each geodesic \bar{C} of (\bar{M}, \bar{g}) is either a horizontal geodesic, or it has no points at which it is tangent to $H\bar{M}$.* According to the terminology from §4, we call \bar{C} from the latter case, a *totally non-horizontal geodesic* of (\bar{M}, \bar{g}) . Next, we present the equations of these categories of geodesics of (\bar{M}, \bar{g}) . First, we state the following.

Theorem 6.2 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Then any horizontal geodesic \bar{C} of (\bar{M}, \bar{g}) is given by the system:*

$$(a) \quad \frac{d^2 x^\gamma}{dt^2} + \Gamma_\alpha^\gamma{}_\beta(x^\mu) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0, \\ (b) \quad \frac{dx^4}{dt} + A_\alpha(x^i) \frac{dx^\alpha}{dt} = 0. \quad (6.2)$$

Proof. As \bar{C} is a horizontal geodesic of (\bar{M}, \bar{g}) , from (4·13) and (4·14) we deduce (6·2), via (6·1a). ■

Now, by (6·1b) we note that $g_{\alpha\beta}(x^\mu)$ define a Lorentz metric on M which we denote by the same symbol g . Moreover, by using (1·10) and (6·1b) in (3·8), we deduce that the local coefficients $\Gamma_\alpha^\gamma{}_\beta(x^\mu)$ from (6·2a) are given by

$$\Gamma_\alpha^\gamma{}_\beta(x^\mu) = \frac{1}{2} g^{\gamma\nu} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right\}. \quad (6.3)$$

Hence, in this particular case, the base manifold M becomes a spacetime with Lorentz metric g and Levi-Civita connection given by its local coefficients from (6·3). This, together with Theorem 6.2, enables us to state the following interesting result.

Corollary 6.1 *Let (\bar{M}, \bar{g}) be a Kaluza-Klein space with bundle-like metric. Then the motions of the spacetime (M, g) coincide with the horizontal induced motions on M which are obtained by the projection of the horizontal geodesics of (\bar{M}, \bar{g}) on M .*

Remark 6.1 Corollary 6.1 is similar to the assertion (a) of Corollary 5.3 on the horizontal geodesics of the classical Kaluza-Klein space. The difference is that in (6.2b) the functions A_α are not necessarily independent of x^4 . ■

Next, we state the following.

Theorem 6.3 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Then any totally non-horizontal geodesic \bar{C} of (\bar{M}, \bar{g}) is given by the system:*

$$\begin{aligned} \text{(a)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta(x^\mu) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + \varepsilon F_\alpha{}^\gamma(x^i) \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} \\ & + \varepsilon B^\gamma(x^i) \left(\frac{\delta x^4}{\delta t} \right)^2 = 0, \\ \text{(b)} \quad & \frac{d}{dt} \left(\frac{\delta x^4}{\delta t} \right) - B_\alpha \frac{dx^\alpha}{dt} \frac{\delta x^4}{\delta t} = 0, \end{aligned} \tag{6.4}$$

subject to the constraint

$$\frac{dx^4}{dt} + A_\alpha(x^i) \frac{dx^\alpha}{dt} \neq 0, \tag{6.5}$$

at any of its points.

Proof. Since \bar{C} is a totally non-horizontal geodesics in (\bar{M}, \bar{g}) , from Theorem 6.1 we deduce that the constraint (6.5) must be satisfied at any of its points. Then Eq. (6.4) is obtained from (4.6), taking into account (6.1a) and (2.12b). ■

From Theorem 6.3 we deduce the following corollary.

Corollary 6.2 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Then, any totally non-horizontal geodesic \bar{C} of (\bar{M}, \bar{g}) satisfying the constraint*

$$\frac{dx^4}{dt} + A_\alpha(x^i) \frac{dx^\alpha}{dt} = c, \tag{6.6}$$

where c is a non-zero constant, is given by the system:

$$\begin{aligned} \text{(a)} \quad & \frac{d^2x^\gamma}{dt^2} + \Gamma_\alpha{}^\gamma{}_\beta(x^\mu) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + \varepsilon c \left\{ F_\alpha{}^\gamma(x^i) \frac{dx^\alpha}{dt} + c B^\gamma(x^i) \right\} = 0, \\ \text{(b)} \quad & B_\alpha(x^i) \frac{dx^\alpha}{dt} = 0. \end{aligned} \tag{6.7}$$

We close this section with a discussion on the existence of the above two categories of geodesics. First, we prove the following.

Theorem 6.4 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Suppose that $\bar{x}_0 = (\bar{x}_0^i)$ is a point in \bar{M} and $u = u^\alpha(\delta/\delta x^\alpha)$ is a horizontal vector at \bar{x}_0 . Then there exists a unique horizontal geodesic passing through \bar{x}_0 and tangent to u .*

Proof. First, we write the system (6.2), that gives the horizontal geodesics, as follows:

$$\frac{dx^\gamma}{dt} = y^\gamma, \quad \frac{dy^\gamma}{dt} = -\Gamma_\alpha{}^\gamma{}_\beta(x^\mu)y^\alpha y^\beta, \quad \frac{dx^4}{dt} = -A_\alpha(x^i) \frac{dx^\alpha}{dt}. \quad (6.8)$$

Then, by the well known “Existence and Uniqueness Theorem for ODE” (cf. Sternberg,²⁷⁾ p. 372), there exists a unique solution $(x^i(t), y^\alpha(t))$ such that $x^i(0) = \bar{x}^i(0)$ and $y^\alpha(0) = u^\alpha$. ■

The situation is different for totally non-horizontal geodesics of (\bar{M}, \bar{g}) . First, by using (1.18) and (2.4b), we deduce that (6.4b) is equivalent to

$$\frac{d^2x^4}{dt^2} = \left(B_\alpha(x^i)A_\beta(x^i) - \frac{\partial A_\alpha}{\partial x^\beta} \right) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}. \quad (6.9)$$

Then, the system (6.4) can be expressed as follows:

$$\begin{aligned} \frac{dx^\gamma}{dt} &= y^\gamma, & \frac{dx^4}{dt} &= y^4, \\ \frac{dy^\gamma}{dt} &= -\Gamma_\alpha{}^\gamma{}_\beta(x^\mu)y^\alpha y^\beta - \varepsilon F_\alpha{}^\gamma(x^i)y^\alpha(y^4 + A_\beta(x^i)y^\beta) \\ &\quad - \varepsilon B^\gamma(x^i)(y^4 + A_\beta(x^i)y^\beta)^2, \\ \frac{dy^4}{dt} &= \left(B_\alpha(x^i)A_\beta(x^i) - \frac{\partial A_\alpha}{\partial x^\beta} \right) y^\alpha y^\beta. \end{aligned} \quad (6.10)$$

According to the constraint (6.5), the initial conditions (\bar{x}_0^i, u^i) must satisfy

$$u^4 + A_\alpha(\bar{x}_0^i)u^\alpha \neq 0. \quad (6.11)$$

Thus, due to the above theorem on ODE, there exists a unique solution of (6.10) with initial conditions satisfying (6.11). Moreover, by continuity of this solution, the constraint (6.5) is locally satisfied by this solution.

Theorem 6.5 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Suppose that $\bar{x}_0 = (\bar{x}_0^i)$ is a point in \bar{M} and $u = u^i(\partial/\partial x^i)$ is a tangent vector to \bar{M} at \bar{x}_0 satisfying (6.11). Then there exists a unique totally non-horizontal geodesic passing through \bar{x}_0 and tangent to u at \bar{x}_0 .*

Next, we are concerned with the particular class of totally non-horizontal geodesics, which are described in Corollary 6.2. First, we consider four non-zero constants $c_{(\gamma)}$ and four vectors $(u_{(\gamma)}^\alpha)$, $\gamma \in \{0, 1, 2, 3\}$. Then, we suppose that through $\bar{x}_0 = (\bar{x}_0^i)$ there are passing four totally non-horizontal geodesics $\bar{C}_{(\gamma)}$ that are tangent at \bar{x}_0 to the vectors

$$u_{(\gamma)} = (u_{(\gamma)}^\alpha) \frac{\delta}{\delta x^\alpha} \Big|_{\bar{x}_0} + c_{(\gamma)} \frac{\partial}{\partial x^4} \Big|_{\bar{x}_0}, \quad (6.12)$$

and satisfy the system given by (6.6) and (6.7) with c replaced by c_γ . We say that $\bar{C}_{(\gamma)}$ are *linearly independent totally non-horizontal geodesics*, if the vectors $(u_{(\gamma)}^\alpha)$ are linearly independent. Now, we can prove the following surprising result.

Theorem 6.6 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space with bundle-like metric. Suppose that through each point of \bar{M} there are passing four linearly independent totally non-horizontal geodesics satisfying the system given by (6.6) and (6.7). Then (\bar{M}, \bar{g}) must be the classical Kaluza-Klein space, that is, A_α must be functions of (x^0, x^1, x^2, x^3) alone, for all $\alpha \in \{0, 1, 2, 3\}$.*

Proof. Let $\bar{x}_0 = (\bar{x}_0^i)$ be a fixed, but arbitrary point of \bar{M} , and \bar{C}_γ be the four linearly independent geodesics passing through \bar{x}_0 and tangent to vectors $u_{(\gamma)}$ from (6.12). Then, by (6.7b) we must have

$$B_\alpha(\bar{x}_0^i)u_{(\gamma)}^\alpha = 0, \quad \gamma \in \{0, 1, 2, 3\}. \tag{6.13}$$

As the vectors $(u_{(\gamma)}^\alpha)$ are linearly independent, from (6.13) we obtain $B_\alpha(\bar{x}_0^i) = 0$, for all $\alpha \in \{0, 1, 2, 3\}$. Taking into account that \bar{x}_0 is an arbitrary point of \bar{M} , we deduce that B_α are vanishing at any point of \bar{M} . Hence, by (2.4b), we conclude that A_α are functions of (x^0, x^1, x^2, x^3) alone. ■

Finally, we remark that Theorem 6.5 does not state the existence of particular geodesics satisfying the constraint (6.6). However, we have the following.

Theorem 6.7 *Let (\bar{M}, \bar{g}) be the classical Kaluza-Klein space. Then through each point $\bar{x}_0 \in \bar{M}$ is passing a unique totally non-horizontal geodesic that is tangent to the vector*

$$u = u^\alpha \frac{\delta}{\delta x^\alpha} \Big|_{\bar{x}_0} + c \frac{\partial}{\partial x^4} \Big|_{\bar{x}_0}, \quad c \neq 0,$$

and satisfies the constraint (6.6).

Proof. In this case, we have $B_\alpha = 0$, for all $\alpha \in \{0, 1, 2, 3\}$. Hence, the system given by (6.6) and (6.7) can be expressed as follows:

$$\begin{aligned} \frac{dx^\gamma}{dt} &= y^\gamma, & \frac{dy^\gamma}{dt} &= -\Gamma_\alpha{}^\gamma{}_\beta(x^\mu)y^\alpha y^\beta - \varepsilon c F_\alpha{}^\gamma(x^\mu)y^\alpha, \\ \frac{dx^4}{dt} &= -A_\alpha(x^\mu)y^\alpha + c. \end{aligned} \tag{6.14}$$

Thus, the assertion of the theorem is obtained by applying the above theorem on ODE for the system (6.14). ■

§7. Horizontal Ricci tensor and horizontal scalar curvature

Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space and ∇ be the Riemannian horizontal connection on $H\bar{M}$ given by (3.4). We know that ∇ is a metric connection and its torsion tensor field is given by (3.3). In the first part of this section, we present the main properties of the curvature tensor field of ∇ . For this purpose, we need an extension of ∇ from $H\bar{M}$ to $T\bar{M}$. More precisely, we consider a linear connection $\tilde{\nabla}$ on \bar{M} given by

$$(a) \tilde{\nabla}_X hY = \nabla_X hY, \quad (b) \tilde{\nabla}_X vY \in \Gamma(V\bar{M}). \tag{7.1}$$

An example of such extension is given by

$$\tilde{\nabla}_X Y = \nabla_X hY + v\bar{\nabla}_X vY,$$

where $\bar{\nabla}$ is the Levi-Civita connection on (\bar{M}, \bar{g}) .

Now, denote by \tilde{T} and \tilde{R} the torsion and curvature tensor fields of $\tilde{\nabla}$, that is, we have:

$$\begin{aligned} \text{(a)} \quad & \tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y], \\ \text{(b)} \quad & \tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z. \end{aligned} \tag{7.2}$$

Proposition 7.1 *The torsion and curvature tensor fields of an extension $\tilde{\nabla}$ of ∇ satisfy the following:*

$$\begin{aligned} \text{(a)} \quad & \tilde{T}(hX, hY) = F(hX, hY), \\ \text{(b)} \quad & h\tilde{T}(vX, hY) = D(vX, hY), \\ \text{(c)} \quad & h\tilde{T}(vX, vY) = 0, \\ \text{(d)} \quad & \tilde{R}(hX, hY)hZ = R(hX, hY)hZ, \\ \text{(e)} \quad & \tilde{R}(vX, hY)hZ = R(vX, hY)hZ, \\ \text{(f)} \quad & h\tilde{R}(hY, hZ)vX = 0, \end{aligned} \tag{7.3}$$

for all $X, Y, Z \in \Gamma(T\bar{M})$, where F and D are the horizontal tensor fields given by (2.5) and (2.10) respectively, and R is the curvature tensor field of ∇ given by

$$R(X, Y)hZ = \nabla_X \nabla_Y hZ - \nabla_Y \nabla_X hZ - \nabla_{[X, Y]} hZ. \tag{7.4}$$

Proof. First, by using (7.2a), (7.1a), (3.1), (3.3a) and (2.5), we obtain

$$\begin{aligned} \tilde{T}(hX, hY) &= \nabla_{hX} hY - \nabla_{hY} hX - h[hX, hY] - v[hX, hY] \\ &= T(hX, hY) + F(hX, hY) = F(hX, hY), \end{aligned}$$

which proves (7.3a). Similarly, by using (7.2a), (7.1) and (3.4b), we obtain (7.3b).

Next, (7.3c) is obtained from (7.2a), by using (7.1b), and taking into account that $V\bar{M}$ is an integrable distribution. Now, (7.3d) and (7.3e), are deduced by direct calculations in (7.2b), using (7.1a) and (7.4). Finally, (7.3f) is obtained from (7.2b) by using (7.1b). ■

Next, we recall from Kobayashi and Nomizu,¹³⁾ (p. 135), that the torsion and curvature tensor fields of $\tilde{\nabla}$ satisfy the following Bianchi identities:

$$\begin{aligned} \text{(a)} \quad & \sum_{(X, Y, Z)} \{(\tilde{\nabla}_X \tilde{T})(Y, Z) + \tilde{T}(\tilde{T}(X, Y), Z) - \tilde{R}(X, Y)Z\} = 0, \\ \text{(b)} \quad & \sum_{(X, Y, Z)} \{(\tilde{\nabla}_X \tilde{R})(Y, Z)U + \tilde{R}(\tilde{T}(X, Y), Z)U\} = 0, \end{aligned} \tag{7.5}$$

where $\sum_{(X, Y, Z)}$ is a cyclic sum with respect to X, Y, Z . Now, we prove the following.

Theorem 7.1 (Bianchi Identities for Riemannian Horizontal Connection) *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space and ∇ the Riemannian horizontal connection on $H\bar{M}$. Then the curvature tensor field R of ∇ satisfies the following*

identities:

$$\begin{aligned}
 \text{(a)} \quad & \sum_{(hX, hY, hZ)} \{R(hX, hY)hZ - D(F(hX, hY), hZ)\} = 0, \\
 \text{(b)} \quad & \sum_{(hX, hY, hZ)} \{(\nabla_{hX}R)(hY, hZ)hU + R(F(hX, hY), hZ)hU\} = 0, \tag{7.6}
 \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(T\bar{M})$.

Proof. First, by using (7.1), (7.3a), (7.3b), (2.5) and (7.3d), we obtain

$$h(\tilde{\nabla}_{hX}\tilde{T})(hY, hZ) = 0, \quad h\tilde{T}(\tilde{T}(hX, hY), hZ) = D(F(hX, hY), hZ). \tag{7.7}$$

Then, (7.6a) is obtained from (7.5a) by using (7.7) and (7.3d). Next, by using (7.3d), (7.1a), (7.3a) and (7.3e), we deduce that

$$\begin{aligned}
 (\tilde{\nabla}_{hX}\tilde{R})(hY, hZ)hU &= (\nabla_{hX}R)(hY, hZ)hU, \\
 \tilde{R}(\tilde{T}(hX, hY), hZ)hU &= R(F(hX, hY), hZ)hU. \tag{7.8}
 \end{aligned}$$

Thus, by using (7.8) into (7.5b), we obtain (7.6b). ■

Next, we define the curvature tensor field of type (0, 4) of ∇ , which is denoted by the same symbol R and it is given by

$$R(hX, hY, hZ, hU) = g(R(hX, hY)hU, hZ). \tag{7.9}$$

Then, we state the following.

Theorem 7.2 *The curvature tensor field of type (0, 4) of the Riemannian horizontal connection ∇ on $H\bar{M}$ satisfies the identities:*

$$\begin{aligned}
 \text{(a)} \quad & R(hX, hY, hZ, hU) + R(hY, hX, hZ, hU) = 0, \\
 \text{(b)} \quad & R(hX, hY, hZ, hU) + R(hX, hY, hU, hZ) = 0, \\
 \text{(c)} \quad & R(hZ, hX, hY, hU) - R(hY, hU, hZ, hX) \\
 & + \sum_{(hX, hY, hZ, hU)} \{D(hX, hY, F(hZ, hU))\} = 0, \tag{7.10}
 \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(T\bar{M})$, where $\sum_{(hX, hY, hZ, hU)}$ is a cyclic sum with respect to the vectors hX, hY, hZ, hU , that is, we have

$$\begin{aligned}
 \sum_{(hX, hY, hZ, hU)} \{D(hX, hY, F(hZ, hU))\} &= D(hX, hY, F(hZ, hU)) \\
 &+ D(hY, hZ, F(hU, hX)) + D(hZ, hU, F(hX, hY)) \\
 &+ D(hU, hX, F(hY, hZ)). \tag{7.11}
 \end{aligned}$$

Proof. First, (7.10a) is obtained by using (7.9) and taking into account the skew-symmetry of R from (7.4) with respect to the first two variables. Then, taking into account that ∇ is a metric connection on $H\bar{M}$, by direct calculations using (7.9) and (7.4), we obtain (7.10b). Next, by using (7.6a), (7.9), (7.10b) and (2.10), we deduce that

$$\sum_{(hX, hY, hZ)} \{R(hX, hY, hZ, hU) + D(hU, hX, F(hY, hZ))\} = 0. \tag{7.12}$$

Finally, by a combinatorial exercise similar to the Riemannian geometry (cf. Kobayashi and Nomizu,¹³ p. 198), using (7.12), (7.10a), (7.10b), (7.11) and taking into account the skew-symmetry of F , we obtain (7.10c). ■

Now, we are in position to define a tensor field of Ricci type for the Riemannian horizontal connection ∇ on $H\bar{M}$. First, we consider a local orthonormal basis $\{E_\alpha\}$, $\alpha \in \{0, 1, 2, 3\}$, in $\Gamma(H\bar{M})$, and we denote by ε_α the signature of E_α , that is, we have

$$\varepsilon_\alpha = g(E_\alpha, E_\alpha), \quad \alpha \in \{0, 1, 2, 3\}. \tag{7.13}$$

Then, we define the *horizontal Ricci tensor* of the general Kaluza-Klein space (\bar{M}, \bar{g}) , as follows:

$$\text{Ric}(hX, hY) = \sum_{\alpha=0}^3 \varepsilon_\alpha \{R(E_\alpha, hX, E_\alpha, hY)\}, \tag{7.14}$$

for all $X, Y \in \Gamma(T\bar{M})$, where R is the curvature tensor field of ∇ , given by (7.4). Clearly, Ric is a $\mathcal{F}(\bar{M})$ -bilinear mapping on $\Gamma(H\bar{M})$, and, therefore, according to the theory from §2, it is a horizontal tensor field on (\bar{M}, \bar{g}) . However, in general, Ric is not a symmetric horizontal tensor field. Now, we define the *horizontal scalar curvature* \mathbf{R} of (\bar{M}, \bar{g}) by

$$\mathbf{R} = \sum_{\alpha=0}^3 \varepsilon_\alpha \text{Ric}(E_\alpha, E_\alpha). \tag{7.15}$$

We should note that both formulae (7.14) and (7.15) do not depend on the orthonormal basis in $\Gamma(T\bar{M})$. In order to write down the field equations induced by the Einstein equations on (\bar{M}, \bar{g}) , we need to express the main geometric objects and some identities, by using an adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ on (\bar{M}, \bar{g}) .

First, we put

$$\begin{aligned} \text{(a)} \quad R\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right) \frac{\delta}{\delta x^\alpha} &= R_{\alpha}{}^{\mu}{}_{\beta\gamma} \frac{\delta}{\delta x^\mu}, \\ \text{(b)} \quad R\left(\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\beta}\right) \frac{\delta}{\delta x^\alpha} &= R_{\alpha}{}^{\mu}{}_{\beta 4} \frac{\delta}{\delta x^\mu}, \\ \text{(c)} \quad R\left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) &= R_{\alpha\beta\gamma\mu}. \end{aligned} \tag{7.16}$$

Then, by direct calculations, using (7.16), (7.4), (3.7), (3.9), (2.3) and (1.13), we obtain

$$\begin{aligned}
 \text{(a)} \quad R_{\alpha}^{\mu}{}_{\beta\gamma} &= \frac{\delta\Gamma_{\alpha}^{\mu}{}_{\beta}}{\delta x^{\gamma}} - \frac{\delta\Gamma_{\alpha}^{\mu}{}_{\gamma}}{\delta x^{\beta}} + \Gamma_{\alpha}^{\nu}{}_{\beta} \Gamma_{\nu}^{\mu}{}_{\gamma} - \Gamma_{\alpha}^{\nu}{}_{\gamma} \Gamma_{\nu}^{\mu}{}_{\beta} - F_{\beta\gamma} D_{\alpha}^{\mu}, \\
 \text{(b)} \quad R_{\alpha}^{\mu}{}_{\beta 4} &= \frac{\partial\Gamma_{\alpha}^{\mu}{}_{\beta}}{\partial x^4} - D_{\alpha}^{\mu}{}_{|\beta} + D_{\beta} D_{\alpha}^{\mu}, \\
 \text{(c)} \quad R_{\alpha\beta\gamma\mu} &= g_{\beta\nu} R_{\alpha}^{\nu}{}_{\gamma\mu},
 \end{aligned} \tag{7.17}$$

where $D_{\alpha}^{\mu}{}_{|\beta}$ is the horizontal covariant derivative defined by ∇ and given by

$$D_{\alpha}^{\mu}{}_{|\beta} = \frac{\delta D_{\alpha}^{\mu}}{\delta x^{\beta}} - D_{\nu}^{\mu} \Gamma_{\alpha}^{\nu}{}_{\beta} + D_{\alpha}^{\nu} \Gamma_{\nu}^{\mu}{}_{\beta}.$$

Next, we take $hX = \delta/\delta x^{\alpha}$, $hY = \delta/\delta x^{\beta}$ and $hZ = \delta/\delta x^{\gamma}$ into (7.6a), and by using (7.16a), (2.6) and (2.11b), we express the *first Bianchi identity* as follows:

$$\sum_{(\alpha,\beta,\gamma)} \{R_{\alpha}^{\mu}{}_{\beta\gamma} + F_{\alpha\beta} D_{\gamma}^{\mu}\} = 0. \tag{7.18}$$

Similarly, we take $hX = \delta/\delta x^{\alpha}$, $hY = \delta/\delta x^{\beta}$, $hZ = \delta/\delta x^{\gamma}$, $hU = \delta/\delta x^{\mu}$ in (7.6b), and by using (7.16a), (3.7a), (2.6), and (7.16b), we infer that the *second Bianchi identity* is expressed as follows:

$$\sum_{(\alpha,\beta,\gamma)} \{R_{\mu}^{\nu}{}_{\alpha\beta|\gamma} - F_{\alpha\beta} R_{\mu}^{\nu}{}_{\gamma 4}\} = 0, \tag{7.19}$$

where we used the covariant derivative defined by ∇ , that is, we have:

$$R_{\mu}^{\nu}{}_{\alpha\beta|\gamma} = \frac{\delta R_{\mu}^{\nu}{}_{\alpha\beta}}{\delta x^{\gamma}} + R_{\mu}^{\sigma}{}_{\alpha\beta} \Gamma_{\sigma}^{\nu}{}_{\gamma} - R_{\sigma}^{\nu}{}_{\alpha\beta} \Gamma_{\mu}^{\sigma}{}_{\gamma} - R_{\mu}^{\nu}{}_{\sigma\beta} \Gamma_{\alpha}^{\sigma}{}_{\gamma} - R_{\mu}^{\nu}{}_{\alpha\sigma} \Gamma_{\beta}^{\sigma}{}_{\gamma}.$$

Also, we take $hX = \delta/\delta x^{\mu}$, $hY = \delta/\delta x^{\gamma}$, $hZ = \delta/\delta x^{\beta}$ and $hU = \delta/\delta x^{\alpha}$ in (7.10), and by using (7.16c), (2.6) and (2.11a), we obtain the identities:

$$\begin{aligned}
 \text{(a)} \quad R_{\alpha\beta\gamma\mu} + R_{\alpha\beta\mu\gamma} &= 0, \\
 \text{(b)} \quad R_{\alpha\beta\gamma\mu} + R_{\beta\alpha\gamma\mu} &= 0, \\
 \text{(c)} \quad R_{\alpha\gamma\mu\beta} - R_{\mu\beta\alpha\gamma} &= F_{\alpha\beta} D_{\gamma\mu} + F_{\beta\gamma} D_{\mu\alpha} + F_{\gamma\mu} D_{\alpha\beta} + F_{\mu\alpha} D_{\beta\gamma}.
 \end{aligned} \tag{7.20}$$

Now, in order to obtain the local components of the horizontal Ricci tensor with respect to the adapted frame $\{\delta/\delta x^{\alpha}\}$, we consider the local orthonormal frame $\{E_{\alpha}\}$ in $\Gamma(H\bar{M})$, and put

$$E_{\alpha} = E_{\alpha}^{\gamma} \frac{\delta}{\delta x^{\gamma}}. \tag{7.21}$$

Then, we have

$$g^{\gamma\mu} = \sum_{\alpha=0}^3 \varepsilon_{\alpha} E_{\alpha}^{\gamma} E_{\alpha}^{\mu}. \tag{7.22}$$

To show this, we put

$$\frac{\delta}{\delta x^{\beta}} = \bar{E}_{\beta}^{\lambda} E_{\lambda},$$

and, from (1.13), we deduce that

$$g_{\beta\gamma} = \sum_{\lambda=0}^3 \varepsilon_{\lambda} \bar{E}_{\beta}^{\lambda} \bar{E}_{\gamma}^{\lambda}.$$

Then, taking into account that $[\bar{E}_{\beta}^{\lambda}]$ is the inverse matrix of $[E_{\alpha}^{\gamma}]$, we obtain

$$g_{\beta\gamma} g^{\gamma\mu} = \sum_{\lambda, \alpha=0}^3 \varepsilon_{\lambda} \varepsilon_{\alpha} \bar{E}_{\beta}^{\lambda} \bar{E}_{\gamma}^{\lambda} E_{\alpha}^{\gamma} E_{\alpha}^{\mu} = \sum_{\lambda, \alpha=0}^3 \varepsilon_{\lambda} \varepsilon_{\alpha} \delta_{\alpha}^{\lambda} E_{\alpha}^{\mu} \bar{E}_{\beta}^{\lambda} = E_{\alpha}^{\mu} \bar{E}_{\beta}^{\alpha} = \delta_{\beta}^{\mu},$$

which proves (7.22). Next, we put

$$R_{\alpha\beta} = \text{Ric} \left(\frac{\delta}{\delta x^{\beta}}, \frac{\delta}{\delta x^{\alpha}} \right), \tag{7.23}$$

and, by using (7.14), (7.21), (7.22), (7.16c) and (7.17c), we obtain

$$R_{\alpha\beta} = R_{\alpha}^{\mu}{}_{\beta\mu}, \tag{7.24}$$

where the components from the right hand side are given by (7.17a). Also, by using (7.21), (7.22) and (7.23) into (7.15), we deduce that the horizontal scalar curvature has a local expression as in the pseudo-Riemannian geometry:

$$\mathbf{R} = g^{\gamma\mu} R_{\gamma\mu}, \tag{7.25}$$

but the local components in the right part are considered with respect to adapted frames and coframes on $H\bar{M}$.

As the theory we develop in the next section needs a symmetric horizontal Ricci tensor, we state the following.

Theorem 7.3 *The horizontal Ricci tensor of a general Kaluza-Klein space is symmetric, if and only if we have*

$$F_{\alpha}^{\gamma} D_{\gamma\mu} - F_{\mu}^{\gamma} D_{\gamma\alpha} = F_{\alpha\mu} D_{\gamma}^{\gamma}. \tag{7.26}$$

Proof. Contracting (7.20c) with $g^{\gamma\beta}$, and taking into account (7.24) and the skew-symmetry of $F_{\beta\gamma}$, we obtain

$$R_{\alpha\mu} - R_{\mu\alpha} = F_{\alpha}^{\gamma} D_{\gamma\mu} - F_{\mu}^{\gamma} D_{\gamma\alpha} + F_{\mu\alpha} D_{\gamma}^{\gamma}.$$

Thus, Ric is symmetric, if and only if (7.26) is satisfied. ■

In what follows, we suppose that the horizontal Ricci tensor is symmetric. Then, we put

$$R_{\alpha}^{\gamma} = g^{\gamma\beta} R_{\alpha\beta}, \tag{7.27}$$

and consider its covariant derivative with respect to ∇ , that is, we have

$$R_{\alpha|\mu}^{\gamma} = \frac{\delta R_{\alpha}^{\gamma}}{\delta x^{\mu}} + R_{\alpha}^{\nu} \Gamma_{\nu}^{\gamma}{}_{\mu} - R_{\nu}^{\gamma} \Gamma_{\alpha}^{\nu}{}_{\mu}.$$

Also, we put

$$\mathbf{R}_{|\mu} = \frac{\delta \mathbf{R}}{\delta x^\mu}. \tag{7.28}$$

Then, we prove the following.

Proposition 7.2 *The horizontal Ricci tensor and the horizontal scalar curvature of a general Kaluza-Klein space (\bar{M}, \bar{g}) satisfy the identity*

$$2R_{\beta|\alpha}^\alpha - \mathbf{R}_{|\beta} = g^{\mu\alpha} R_{\mu\ \alpha 4}^\nu F_{\beta\nu} + F_\nu^\mu R_{\mu\ \beta 4}^\nu - F_{\beta}^\mu R_{\mu\ \nu 4}^\nu. \tag{7.29}$$

Proof. Contracting the indices ν and γ in (7.19) and using (7.24), we obtain

$$R_{\mu\ \alpha\beta|\nu}^\nu + R_{\mu\beta|\alpha} - R_{\mu\alpha|\beta} = R_{\mu\ \alpha 4}^\nu F_{\beta\nu} - F_{\alpha\nu} R_{\mu\ \beta 4}^\nu + F_{\alpha\beta} R_{\mu\ \nu 4}^\nu. \tag{7.30}$$

Next, by using (7.17c), (3.11a), (7.20a), (7.20b), (7.24) and (7.27), we infer that

$$g^{\mu\alpha} R_{\mu\ \alpha\beta|\nu}^\nu = -g^{\mu\alpha} g^{\nu\lambda} R_{\lambda\mu\alpha\beta|\nu} = g^{\nu\lambda} R_{\lambda\ \beta\mu|\nu}^\mu = R_{\beta|\nu}^\nu. \tag{7.31}$$

Finally, contracting (7.30) by $g^{\mu\alpha}$ and using (7.31), (3.11a), (7.27), (7.25) and (3.12a), we obtain (7.29). ■

Now, suppose that on (\bar{M}, \bar{g}) the following constraint is satisfied

$$F_\nu^\mu R_{\mu\ \beta 4}^\nu - F_{\beta}^\mu R_{\mu\ \nu 4}^\nu = g^{\mu\alpha} R_{\mu\ \alpha 4}^\nu F_{\nu\beta}. \tag{7.32}$$

We call (\bar{M}, \bar{g}) satisfying (7.26) and (7.32) a *relativistic general Kaluza-Klein space*. Then, taking into account Theorem 7.3 and Proposition 7.2, we state the following.

Proposition 7.3 *Let (\bar{M}, \bar{g}) be a relativistic general Kaluza-Klein space. Then the horizontal Ricci tensor is symmetric and satisfies*

$$2R_{\beta|\alpha}^\alpha - \mathbf{R}_{|\beta} = 0, \quad \forall \beta \in \{0, 1, 2, 3\}. \tag{7.33}$$

In the next proposition we present two large classes of relativistic general Kaluza-Klein spaces.

Proposition 7.4 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space satisfying one of the following conditions:*

- (i) (\bar{M}, \bar{g}) has bundle-like metric.
- (ii) The horizontal electromagnetic tensor field vanishes identically on \bar{M} .

Then (\bar{M}, \bar{g}) is a relativistic general Kaluza-Klein space.

Proof. First, suppose that (\bar{M}, \bar{g}) has bundle-like metric. Then, (6.1a) implies (7.26). Also, by using (6.1a) and (6.3) into (7.17b), we obtain $R_{\alpha\ \beta 4}^\mu = 0$, for all $\alpha, \beta, \mu \in \{0, 1, 2, 3\}$. Thus, (7.32) is satisfied too. In case of condition (ii), we have $F_{\alpha\beta} = 0$, for all $\alpha, \beta \in \{0, 1, 2, 3\}$, and therefore both conditions (7.26) and (7.32) are satisfied. ■

Remark 7.1 Clearly, the model presented by Randall and Sundrum^{(23),(24)} as an alternative to compactification satisfies condition (ii) and, therefore, it is a relativistic general Kaluza-Klein space. ■

§8. The horizontal Einstein gravitational tensor field

In the first part of this section we define some new differential operators related to the horizontal distribution. First, we consider a smooth function f on \bar{M} , and define its *horizontal differential* as the horizontal 1-form denoted by $d_h f$ and defined by

$$d_h f = \frac{\delta f}{\delta x^\alpha} dx^\alpha = f|_\alpha dx^\alpha. \tag{8.1}$$

Then, we consider an adapted coframe field $\{dx^\alpha, \delta x^4\}$, where δx^4 is given by (1.18), and state that

$$d_h f = df - \frac{\partial f}{\partial x^4} \delta x^4, \tag{8.2}$$

where df is the usual differential of f on \bar{M} .

Indeed, by using (1.10) and (1.18) in (8.1), we obtain

$$d_h f = \left(\frac{\partial f}{\partial x^\alpha} - A_\alpha \frac{\partial f}{\partial x^4} \right) dx^\alpha = \frac{\partial f}{\partial x^\alpha} dx^\alpha - \frac{\partial f}{\partial x^4} (\delta x^4 - dx^4) = df - \frac{\partial f}{\partial x^4} \delta x^4.$$

Next, we consider a symmetric horizontal tensor field A of type $(0, 2)$ on \bar{M} , and define its *horizontal divergence*, as the horizontal covector field $\text{div}_h A$ given by

$$(\text{div}_h A)(hY) = \sum_{\gamma=0}^3 \varepsilon_\gamma (\nabla_{E_\gamma} A)(E_\gamma, hY), \quad \forall Y \in \Gamma(T\bar{M}), \tag{8.3}$$

where $\{E_\gamma\}$ is a local orthonormal basis in $\Gamma(H\bar{M})$ and ∇ is the Riemannian horizontal connection given by (3.4). Clearly, (8.3) is independent of the basis $\{E_\gamma\}$ and define a horizontal covector field on \bar{M} . Moreover, (8.3) coincides with the well known formula for the divergence of a symmetric tensor field on a pseudo-Riemannian manifold (cf. O’Neill,¹⁸ p. 86), but here the Riemannian horizontal connection is used instead of the Levi-Civita connection. For our purpose, we need a formula for horizontal divergence by using the local components of A with respect to an adapted frame field $\{\delta/\delta x^\alpha\}$ on $H\bar{M}$. Thus, we put

$$(a) \ A_{\alpha\beta} = A \left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right), \quad (b) \ A_\alpha^\gamma = A_{\alpha\beta} g^{\beta\gamma}, \tag{8.4}$$

and prove the following.

Proposition 8.1 *Let (\bar{M}, \bar{g}) be a general Kaluza-Klein space and A be a symmetric horizontal tensor field on \bar{M} . Then the horizontal divergence of A is given by*

$$\text{div}_h A = A_{\alpha|\gamma}^\gamma dx^\alpha, \tag{8.5}$$

where “ $|$ ” denotes the covariant derivative with respect to ∇ , that is, we have

$$A_{\alpha|\mu}^\gamma = \frac{\delta A_\alpha^\gamma}{\delta x^\mu} + A_\alpha^\nu \Gamma_{\nu\mu}^\gamma - A_\nu^\gamma \Gamma_{\alpha\mu}^\nu.$$

Proof. First, by using (7.21) and (7.22) into (8.3), we obtain

$$(\operatorname{div}_h A) \left(\frac{\delta}{\delta x^\alpha} \right) = g^{\gamma\mu} \left(\nabla_{\frac{\delta}{\delta x^\mu}} A \right) \left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\alpha} \right) = g^{\gamma\mu} A_{\alpha\gamma|\mu}, \tag{8.6}$$

where we have

$$A_{\alpha\gamma|\mu} = \frac{\delta A_{\alpha\gamma}}{\delta x^\mu} - A_{\nu\gamma} \Gamma_{\alpha\ \mu}^\nu - A_{\alpha\nu} \Gamma_{\gamma\ \mu}^\nu.$$

Then, by using (3.11a) and (8.4b), we deduce that

$$(\operatorname{div}_h A) \left(\frac{\delta}{\delta x^\alpha} \right) = A^\mu_{\alpha|\mu},$$

which proves (8.5). ■

Now, following Einstein’s theory of general relativity on a Lorentz manifold, we define on the Lorentz bundle $(H\bar{M}, g)$, the horizontal tensor field G given by

$$G(hX, hY) = \operatorname{Ric}(hX, hY) - \frac{\mathbf{R}}{2} g(hX, hY), \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{8.7}$$

Then, we call G the *horizontal Einstein gravitational tensor field* on the general Kaluza-Klein space (\bar{M}, \bar{g}) . Locally, we put

$$G_{\alpha\beta} = G \left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha} \right), \tag{8.8}$$

and, by using (7.24) and (1.13), we deduce that

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{\mathbf{R}}{2} g_{\alpha\beta}. \tag{8.9}$$

Next, we can state the following important result.

Theorem 8.1 *Let (\bar{M}, \bar{g}) be a relativistic general Kaluza-Klein space. Then the horizontal Einstein gravitational tensor field is a symmetric horizontal tensor field and its horizontal divergence vanishes identically on \bar{M} .*

Proof. First, by Proposition 7.3 and (8.7), we deduce that G is a symmetric horizontal tensor field. Then, from (8.7), we obtain

$$\operatorname{div}_h G = \operatorname{div}_h \operatorname{Ric} - \frac{1}{2} d_h \mathbf{R},$$

since $\operatorname{div}_h g = 0$. Finally, by using (8.5), (8.1) and (7.33), we deduce that

$$\operatorname{div}_h G = \left(R^\gamma_{\alpha|\gamma} - \frac{1}{2} \mathbf{R}_{|\alpha} \right) dx^\alpha = 0.$$

This completes the proof of the theorem. ■

So far, the study of a general Kaluza-Klein space $\bar{M} = M \times K$ was mainly concerned with two particular cases. One of them refers to the so-called factorizable

case, which assumes that on M there exists a Lorentz metric and K is a warped manifold. For a study of some multidimensional cosmological models which fall into the factorizable case, see Günter, Starobinsky and Zhuk.⁹⁾ The nonfactorizable case is due to Randall and Sundrum,^{23),24)} and assumes that M is warped manifold with a factor that depends on the fifth dimension. Also, the canonical metric (4.11) from the space-time-matter theory (see Chapter 6 in Wesson²⁸⁾) belongs to the nonfactorizable case. *It is worth mentioning that all these models are relativistic general Kaluza-Klein spaces and, therefore, their horizontal Einstein gravitational tensor fields are symmetric and of horizontal divergence zero.*

§9. Ricci tensor and scalar curvature of a general Kaluza-Klein space

In the present section, we find the local components of the Ricci tensor of a general Kaluza-Klein space with respect to an adapted frame field. The formulae for both the Ricci tensor and the scalar curvature of (\bar{M}, \bar{g}) , are expressed in terms of the horizontal Ricci tensor, horizontal scalar curvature, and the horizontal tensor fields D, F and B from §2.

Let $\bar{\nabla}$ be the Levi-Civita connection on (\bar{M}, \bar{g}) and \bar{R} be the curvature tensor field of $\bar{\nabla}$ given by a similar formula as in (7.2b). Consider the adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ on \bar{M} and introduce the following local components of \bar{R} :

$$\begin{aligned} \text{(a)} \quad \bar{R} \left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta} \right) \frac{\delta}{\delta x^\alpha} &= \bar{R}_\alpha{}^\mu{}_{\beta\gamma} \frac{\delta}{\delta x^\mu} + \bar{R}_\alpha{}^4{}_{\beta\gamma} \frac{\partial}{\partial x^4}, \\ \text{(b)} \quad v\bar{R} \left(\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\beta} \right) \frac{\delta}{\delta x^\alpha} &= \bar{R}_\alpha{}^4{}_{\beta 4} \frac{\partial}{\partial x^4}. \end{aligned} \tag{9.1}$$

Remark 9.1 It is important to note that the functions $\bar{R}_\alpha{}^\mu{}_{\beta\gamma}$, $\bar{R}_\alpha{}^4{}_{\beta\gamma}$ and $\bar{R}_\alpha{}^4{}_{\beta 4}$ satisfy some tensorial transformations with respect to the four- and cut-transformations (1.7), and therefore they define horizontal tensor fields of type (1, 3), (0, 3) and (0, 2), respectively. This is one of the advantages of our new approach. ■

Now, we prove the following.

Proposition 9.1 *The local components of \bar{R} expressed in (9.1) are given by*

$$\begin{aligned} \text{(a)} \quad \bar{R}_\alpha{}^\mu{}_{\beta\gamma} &= R_\alpha{}^\mu{}_{\beta\gamma} + \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) \left(D_\gamma{}^\mu + \frac{\varepsilon}{2} F_\gamma{}^\mu \right) \\ &\quad - \left(\frac{1}{2} F_{\alpha\gamma} - \varepsilon D_{\alpha\gamma} \right) \left(D_\beta{}^\mu + \frac{\varepsilon}{2} F_\beta{}^\mu \right) - \frac{\varepsilon}{2} F_\alpha{}^\mu F_{\beta\gamma}, \\ \text{(b)} \quad \bar{R}_\alpha{}^4{}_{\beta\gamma} &= \frac{1}{2} (F_{\alpha\beta|\gamma} - F_{\alpha\gamma|\beta}) + \varepsilon (D_{\alpha\gamma|\beta} - D_{\alpha\beta|\gamma}) + B_\alpha F_{\beta\gamma}, \\ \text{(c)} \quad \bar{R}_\alpha{}^4{}_{\beta 4} &= B_{\alpha|\beta} + \frac{1}{2} F_{\alpha\beta|4} - \varepsilon D_{\alpha\beta|4} + F_{\alpha\mu} D_\beta{}^\mu - \varepsilon D_{\alpha\mu} D_\beta{}^\mu \\ &\quad - \frac{\varepsilon}{4} F_\alpha{}^\mu F_{\mu\beta} - B_\alpha B_\beta, \end{aligned} \tag{9.2}$$

where $R_\alpha{}^\mu{}_{\beta\gamma}$ are given by (7.17a), and the covariant derivatives are taken with respect to the Riemannian horizontal connection ∇ and are given by (3.26) and (3.27).

Proof. First, by using (3·13a) and (3·13c), we obtain

$$\begin{aligned} & \overline{\nabla}_{\frac{\delta}{\delta x^\gamma}} \overline{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} \\ &= \left\{ \frac{\delta F_{\alpha^\mu \beta}}{\delta x^\gamma} + \Gamma_{\alpha^\nu \beta} \Gamma_{\nu^\mu \gamma} + \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) \left(D_{\gamma^\mu} + \frac{\varepsilon}{2} F_{\gamma^\mu} \right) \right\} \frac{\delta}{\delta x^\mu} \\ &+ \left\{ \frac{1}{2} \frac{\delta F_{\alpha\beta}}{\delta x^\gamma} + \frac{1}{2} F_{\nu\gamma} \Gamma_{\alpha^\nu \beta} - \varepsilon \frac{\delta D_{\alpha\beta}}{\delta x^\gamma} - \varepsilon D_{\nu\gamma} \Gamma_{\alpha^\nu \beta} \right\} \frac{\partial}{\partial x^4}. \end{aligned} \quad (9.3)$$

Then, by using (2·3a) and (3·13b), we infer that

$$\overline{\nabla}_{\left[\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta} \right]} \frac{\delta}{\delta x^\alpha} = F_{\beta\gamma} \left(D_{\alpha^\mu} + \frac{\varepsilon}{2} F_{\alpha^\mu} \right) \frac{\delta}{\delta x^\mu} - B_\alpha F_{\beta\gamma} \frac{\partial}{\partial x^4}. \quad (9.4)$$

Now, by using (7·2b) for \overline{R} and taking into account (9·3), (9·4), (7·17a), (3·26a) and (3·26b), we deduce that

$$\begin{aligned} \overline{R} \left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta} \right) \frac{\delta}{\delta x^\alpha} &= \left\{ R_{\alpha^\mu \beta\gamma} + \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) \left(D_{\gamma^\mu} + \frac{\varepsilon}{2} F_{\gamma^\mu} \right) \right. \\ &- \left. \left(\frac{1}{2} F_{\alpha\gamma} - \varepsilon D_{\alpha\gamma} \right) \left(D_{\beta^\mu} + \frac{\varepsilon}{2} F_{\beta^\mu} \right) - \frac{\varepsilon}{2} F_{\alpha^\mu} F_{\beta\gamma} \right\} \frac{\delta}{\delta x^\mu} \\ &+ \left\{ \frac{1}{2} (F_{\alpha\beta|\gamma} - F_{\alpha\gamma|\beta}) + \varepsilon (D_{\alpha\gamma|\beta} - D_{\alpha\beta|\gamma}) + B_\alpha F_{\beta\gamma} \right\} \frac{\partial}{\partial x^4}. \end{aligned} \quad (9.5)$$

By comparing (9·1a) and (9·5), we obtain both (9·2a) and (9·2b). Next, by direct calculations using (3·13) and (2·3b), we obtain

$$\begin{aligned} v \overline{\nabla}_{\frac{\partial}{\partial x^4}} \overline{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} &= \left\{ \frac{1}{2} \frac{\partial F_{\alpha\beta}}{\partial x^4} - \varepsilon \frac{\partial D_{\alpha\beta}}{\partial x^4} - B_\mu \Gamma_{\alpha^\mu \beta} \right\} \frac{\partial}{\partial x^4}, \\ v \overline{\nabla}_{\frac{\delta}{\delta x^\beta}} \overline{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} &= \left\{ \left(D_{\alpha^\nu} + \frac{\varepsilon}{2} F_{\alpha^\nu} \right) \left(\frac{1}{2} F_{\nu\beta} - \varepsilon D_{\nu\beta} \right) - \frac{\delta B_\alpha}{\delta x^\beta} \right\} \frac{\partial}{\partial x^4}, \\ v \overline{\nabla}_{\left[\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\beta} \right]} \frac{\delta}{\delta x^\alpha} &= B_\alpha B_\beta \frac{\partial}{\partial x^4}. \end{aligned} \quad (9.6)$$

Then, by using (9·6), (3·26c), (3·27a) and (3·27b) into (7·2b) for \overline{R} , we deduce that

$$\begin{aligned} v \overline{R} \left(\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^\beta} \right) \frac{\delta}{\delta x^\alpha} &= \left\{ B_{\alpha|\beta} + \frac{1}{2} F_{\alpha\beta|4} - \varepsilon D_{\alpha\beta|4} + F_{\alpha\nu} D_{\beta^\nu} - \varepsilon D_{\alpha\nu} D_{\beta^\nu} \right. \\ &- \left. \frac{\varepsilon}{4} F_{\alpha^\nu} F_{\nu\beta} - B_\alpha B_\beta \right\} \frac{\partial}{\partial x^4}. \end{aligned}$$

Finally, comparing this with (9·1b), we obtain (9·2c). This completes the proof of the proposition. ■

Next, we consider the curvature tensor field of type (0,4) of the Levi-Civita connection $\overline{\nabla}$ on $(\overline{M}, \overline{g})$ given by

$$\overline{R}(X, Y, Z, U) = \overline{g}(\overline{R}(X, Y)U, Z), \quad \forall X, Y, Z, U \in \Gamma(T\overline{M}), \quad (9.7)$$

where \bar{R} from the right-hand side is given by (7.2b), with $\tilde{\nabla} = \bar{\nabla}$. Then, following O'Neill,¹⁸⁾ (p. 87), we consider the Ricci tensor of (\bar{M}, \bar{g}) given by

$$\bar{\text{Ric}}(X, Y) = \sum_{\alpha=0}^3 \{ \varepsilon_{\alpha} \bar{R}(E_{\alpha}, X, E_{\alpha}, Y) \} + \varepsilon \bar{R} \left(\frac{\partial}{\partial x^4}, X, \frac{\partial}{\partial x^4}, Y \right), \quad (9.8)$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\{E_{\alpha}\}$ is an orthonormal basis in $\Gamma(H\bar{M})$ of signatures $\{\varepsilon_{\alpha}\}$, and $\partial/\partial x^4$ is the unit vertical vector field of signature ε (cf. (1.6)).

Now, take an adapted frame field $\{\delta/\delta x^{\alpha}, \partial/\partial x^4\}$, and put

$$\begin{aligned} \text{(a)} \quad \bar{R}_{\alpha\beta} &= \bar{\text{Ric}} \left(\frac{\delta}{\delta x^{\beta}}, \frac{\delta}{\delta x^{\alpha}} \right), & \text{(b)} \quad \bar{R}_{\alpha 4} &= \bar{\text{Ric}} \left(\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial x^4} \right), \\ \text{(c)} \quad \bar{R}_{44} &= \bar{\text{Ric}} \left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4} \right). \end{aligned} \quad (9.9)$$

Then, by using (9.9), (9.8), (7.21), (7.22), (9.7), (9.1), (1.13) and (1.6), we obtain

$$\begin{aligned} \text{(a)} \quad \bar{R}_{\alpha\beta} &= g^{\mu\nu} \bar{g} \left(\bar{R} \left(\frac{\delta}{\delta x^{\mu}}, \frac{\delta}{\delta x^{\beta}} \right) \frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\nu}} \right) \\ &+ \varepsilon \bar{g} \left(\bar{R} \left(\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^{\beta}} \right) \frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial x^4} \right) = \bar{R}_{\alpha}{}^{\mu}{}_{\beta\mu} + \bar{R}_{\alpha}{}^4{}_{\beta 4}, \\ \text{(b)} \quad \bar{R}_{\alpha 4} &= g^{\mu\nu} \bar{g} \left(\bar{R} \left(\frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\nu}} \right) \frac{\delta}{\delta x^{\mu}}, \frac{\partial}{\partial x^4} \right) = \varepsilon g^{\mu\nu} \bar{R}_{\mu}{}^4{}_{\nu\alpha}, \\ \text{(c)} \quad \bar{R}_{44} &= g^{\mu\nu} \bar{g} \left(\bar{R} \left(\frac{\partial}{\partial x^4}, \frac{\delta}{\delta x^{\nu}} \right) \frac{\delta}{\delta x^{\mu}}, \frac{\partial}{\partial x^4} \right) = \varepsilon g^{\mu\nu} \bar{R}_{\mu}{}^4{}_{\nu 4}. \end{aligned} \quad (9.10)$$

Next, we prove the following.

Theorem 9.1 *The local components of the Ricci tensor of the general Kaluza-Klein space (\bar{M}, \bar{g}) with respect to the adapted frame field $\{\delta/\delta x^{\alpha}, \partial/\partial x^4\}$ are given by*

$$\begin{aligned} \text{(a)} \quad \bar{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{\varepsilon}{2} F_{\alpha}{}^{\mu} F_{\beta\mu} + \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) D_{\mu}{}^{\mu} + \frac{1}{2} F_{\alpha\mu} D_{\beta}{}^{\mu} \\ &+ \frac{1}{2} D_{\alpha\mu} F_{\beta}{}^{\mu} + B_{\alpha|\beta} + \frac{1}{2} F_{\alpha\beta|4} - \varepsilon D_{\alpha\beta|4} - B_{\alpha} B_{\beta}, \\ \text{(b)} \quad \bar{R}_{\alpha 4} &= \varepsilon \left(\frac{1}{2} F_{\alpha}{}^{\mu}{}_{|\mu} + B^{\mu} F_{\mu\alpha} \right) + D_{\alpha}{}^{\mu}{}_{|\mu} - D_{\mu}{}^{\mu}{}_{|\alpha}, \\ \text{(c)} \quad \bar{R}_{44} &= \varepsilon (B^{\mu}{}_{|\mu} - B^{\mu} B_{\mu}) - D_{\mu}{}^{\mu}{}_{|4} - D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \end{aligned} \quad (9.11)$$

where $R_{\alpha\beta}$ are the local components of the horizontal Ricci tensor given by (7.24).

Proof. First, by using (9.10a), (9.2a), (9.2c), (7.24), and taking into account that F and D are skew-symmetric and symmetric horizontal tensor fields respectively, we obtain (9.11a). Then, (9.11b) is obtained from (9.10b), by using (9.2b), (3.11a) and (3.12). Finally, by using (9.10c), (9.2c), (3.11a) and (3.12), and taking into account that

$$F_{\mu}{}^{\nu} D_{\nu}{}^{\mu} = F_{\mu\gamma} D^{\mu\gamma} = 0,$$

we obtain (9.11c). Thus, the proof is complete. ■

Now, we consider the scalar curvature $\bar{\mathbf{R}}$ of (\bar{M}, \bar{g}) given by

$$\bar{\mathbf{R}} = \sum_{\alpha=0}^3 \varepsilon_\gamma \bar{\text{Ric}}(E_\gamma, E_\gamma) + \varepsilon \bar{\text{Ric}}\left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}\right). \tag{9.12}$$

Then, by using (7.21), (7.22), (9.9a) and (9.9c) into (9.12), we deduce that

$$\bar{\mathbf{R}} = g^{\alpha\beta} \bar{R}_{\alpha\beta} + \varepsilon \bar{R}_{44}. \tag{9.13}$$

Theorem 9.2 *The scalar curvature and the horizontal scalar curvature of the general Kaluza-Klein space (\bar{M}, \bar{g}) are related by*

$$\bar{\mathbf{R}} = \mathbf{R} - \varepsilon \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu^\mu)^2 + 2D_\mu^\mu|_4 + D_\mu^\nu D_\nu^\mu \right\} + 2B^\mu|_\mu - 2B^\mu B_\mu. \tag{9.14}$$

Proof. First, by using (9.11a), (7.25), (3.11) and (3.12), and taking into account that F is a skew-symmetric horizontal tensor field, we obtain

$$g^{\alpha\beta} \bar{R}_{\alpha\beta} = \mathbf{R} - \varepsilon \left\{ \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + (D_\mu^\mu)^2 + D_\mu^\mu|_4 \right\} + B^\mu|_\mu - B^\mu B_\mu. \tag{9.15}$$

Then, (9.14) is obtained from (9.13) by using (9.15) and (9.11c). ■

§10. Horizontal Einstein equations on a relativistic general Kaluza-Klein space

In this section, we suppose that (\bar{M}, \bar{g}) is a relativistic general Kaluza-Klein space, that is, the two constraints (7.26) and (7.32) are satisfied. By Theorem 8.1 we know that the horizontal Einstein gravitational tensor field $G_{\alpha\beta}$ given by (8.9) is a symmetric horizontal tensor field whose horizontal divergence vanishes identically on \bar{M} .

Now, let \bar{G} be the *Einstein gravitational tensor field* of (\bar{M}, \bar{g}) given by

$$\bar{G}(X, Y) = \bar{\text{Ric}}(X, Y) - \frac{\bar{\mathbf{R}}}{2} \bar{g}(X, Y), \quad \forall X, Y \in \Gamma(T\bar{M}), \tag{10.1}$$

and put

$$\begin{aligned} \text{(a)} \quad \bar{G}_{\alpha\beta} &= \bar{G}\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right), & \text{(b)} \quad \bar{G}_{\alpha 4} &= \bar{G}\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4}\right), \\ \text{(c)} \quad \bar{G}_{44} &= \bar{G}\left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}\right). \end{aligned} \tag{10.2}$$

It is easy to see that $(\bar{G}_{\alpha\beta})$ and $(\bar{G}_{\alpha 4})$ define a symmetric horizontal tensor field of type $(0, 2)$ and a horizontal covector field respectively, while \bar{G}_{44} is a function that is globally defined on \bar{M} . Moreover, by using (10.1), (10.2), (9.9), (1.13), (1.14) and (1.6), we obtain

$$\text{(a)} \quad \bar{G}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{\bar{\mathbf{R}}}{2} g_{\alpha\beta}, \quad \text{(b)} \quad \bar{G}_{\alpha 4} = \bar{R}_{\alpha 4}, \quad \text{(c)} \quad \bar{G}_{44} = \bar{R}_{44} - \frac{\varepsilon \bar{\mathbf{R}}}{2}. \tag{10.3}$$

Next, by using the horizontal electromagnetic tensor field $F_{\alpha\beta}$ given by (2.4a), we construct the horizontal tensor field E with the local components

$$E_{\alpha\beta} = E \left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha} \right) = \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu} - F_\alpha{}^\mu F_{\beta\mu}, \tag{10.4}$$

where $F_\alpha{}^\mu$ and $F^{\mu\nu}$ are given by (3.12a) and (3.12b), respectively. Then, according to the terminology from the classical electromagnetic theory, we call E given by (10.4) the *horizontal electromagnetic energy-momentum tensor field* on the relativistic general Kaluza-Klein space (\bar{M}, \bar{g}) .

Proposition 10.1 *The Einstein gravitational tensor field \bar{G} of the relativistic general Kaluza-Klein space (\bar{M}, \bar{g}) is given by*

$$\begin{aligned} \text{(a)} \quad \bar{G}_{\alpha\beta} &= G_{\alpha\beta} + \frac{\varepsilon}{2} E_{\alpha\beta} + \left(\frac{1}{2} F_{\alpha\beta} - \varepsilon D_{\alpha\beta} \right) D_\mu{}^\mu + \frac{1}{2} F_{\alpha\mu} D_\beta{}^\mu \\ &\quad + \frac{1}{2} D_{\alpha\mu} F_\beta{}^\mu + B_{\alpha|\beta} + \frac{1}{2} F_{\alpha\beta|4} - \varepsilon D_{\alpha\beta|4} - B_\alpha B_\beta \\ &\quad + g_{\alpha\beta} \left\{ B^\mu B_\mu - B^\mu{}_{|\mu} + \frac{\varepsilon}{2} \left((D_\mu{}^\mu)^2 + 2D_\mu{}^\mu{}_{|4} + D_\mu{}^\nu D_\nu{}^\mu \right) \right\}, \\ \text{(b)} \quad \bar{G}_{\alpha 4} &= \varepsilon (F_{\alpha\mu}{}_{|\mu} + B^\mu F_{\mu\alpha}) + D_{\alpha\mu}{}_{|\mu} - D_\mu{}^\mu{}_{|\alpha}, \\ \text{(c)} \quad \bar{G}_{44} &= \frac{3}{8} F^{\mu\nu} F_{\mu\nu} - \frac{\varepsilon}{2} \mathbf{R} + \frac{1}{2} \left((D_\mu{}^\mu)^2 - D_\mu{}^\nu D_\nu{}^\mu \right), \end{aligned} \tag{10.5}$$

where $G_{\alpha\beta}$ and \mathbf{R} are the local components of the horizontal Einstein gravitational tensor field and the horizontal scalar curvature of (\bar{M}, \bar{g}) .

Proof. First, by using (9.11a) and (9.14) into (10.3a), and taking into account (8.9) and (10.4), we obtain (10.5a). Then, (10.5b) is obtained from (10.3b) by using (9.11b). Finally, by using (9.11c) and (9.14) into (10.3c), we deduce (10.5c). ■

Next, we suppose that (\bar{M}, \bar{g}) is a relativistic general Kaluza-Klein space containing matter with *stress-energy tensor* \bar{T} . Then, the Einstein equations in (\bar{M}, \bar{g}) have the form

$$\bar{G} = \bar{k} \bar{T}, \tag{10.6}$$

where \bar{k} is a real constant. We consider the adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ on \bar{M} , and put

$$\begin{aligned} \text{(a)} \quad \bar{T}_{\alpha\beta} &= \bar{T} \left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha} \right), & \text{(b)} \quad \bar{T}_{\alpha 4} &= \bar{T} \left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^4} \right), \\ \text{(c)} \quad \bar{T}_{44} &= \bar{T} \left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4} \right). \end{aligned} \tag{10.7}$$

As in the case of the Einstein gravitational tensor field, we note that $\bar{T}_{\alpha\beta}$ and $\bar{T}_{\alpha 4}$ define a symmetric horizontal tensor field and a horizontal covector field respectively, while \bar{T}_{44} is a function globally defined on \bar{M} . Locally, (10.6) are expressed as follows:

$$\text{(a)} \quad \bar{G}_{\alpha\beta} = \bar{k} \bar{T}_{\alpha\beta}, \quad \text{(b)} \quad \bar{G}_{\alpha 4} = \bar{k} \bar{T}_{\alpha 4}, \quad \text{(c)} \quad \bar{G}_{44} = \bar{k} \bar{T}_{44}. \tag{10.8}$$

Now, we can state the following theorem.

Theorem 10.1 *The Einstein equations on the relativistic general Kaluza-Klein space (\bar{M}, \bar{g}) with stress-energy tensor \bar{T} are expressed as follows:*

$$\begin{aligned}
 \text{(a)} \quad G_{\alpha\beta} &= \bar{k}\bar{T}_{\alpha\beta} - \frac{\varepsilon}{2} E_{\alpha\beta} + \left(\varepsilon D_{\alpha\beta} - \frac{1}{2} F_{\alpha\beta} \right) D_{\mu}{}^{\mu} - \frac{1}{2} F_{\alpha\mu} D_{\beta}{}^{\mu} \\
 &\quad - \frac{1}{2} D_{\alpha\mu} F_{\beta}{}^{\mu} - B_{\alpha|\beta} - \frac{1}{2} F_{\alpha\beta|4} + \varepsilon D_{\alpha\beta|4} + B_{\alpha} B_{\beta} \\
 &\quad - g_{\alpha\beta} \left\{ B^{\mu} B_{\mu} - B^{\mu}{}_{|\mu} + \frac{\varepsilon}{2} \left((D_{\mu}{}^{\mu})^2 + 2D_{\mu}{}^{\mu}{}_{|4} + D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} \right) \right\}, \\
 \text{(b)} \quad \frac{1}{2} F_{\alpha}{}^{\mu}{}_{|\mu} + B^{\mu} F_{\mu\alpha} &= \varepsilon (\bar{k}\bar{T}_{\alpha 4} - D_{\alpha}{}^{\mu}{}_{|\mu} + D_{\mu}{}^{\mu}{}_{|\alpha}), \\
 \text{(c)} \quad \frac{3}{8} F^{\mu\nu} F_{\mu\nu} &= \bar{k} \left(\bar{T}_{44} - \frac{\varepsilon}{2} \bar{T}_{\alpha\beta} g^{\alpha\beta} \right) \\
 &\quad + \frac{3}{2} \{ D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} + D_{\mu}{}^{\mu}{}_{|4} + \varepsilon (B^{\mu} B_{\mu} - B^{\mu}{}_{|\mu}) \}. \tag{10-9}
 \end{aligned}$$

Proof. First, (10-9a) is obtained from (10-8a) by using (10-5a). Then, in a similar way, by using (10-5b) into (10-8b), we deduce (10-9b). Next, by contracting (10-4) with $g^{\alpha\beta}$, we obtain

$$g^{\alpha\beta} E_{\alpha\beta} = 0. \tag{10-10}$$

Then, contracting (10-9a) by $g^{\alpha\beta}$ and using (10-10), (8-9), (7-25) and (3-11b), and taking into account that $F_{\alpha\beta}$ are skew-symmetric, we deduce that the horizontal scalar curvature of (\bar{M}, \bar{g}) is given by

$$\mathbf{R} = -\bar{k}\bar{T}_{\alpha\beta} g^{\alpha\beta} + \varepsilon (D_{\mu}{}^{\mu})^2 + 2\varepsilon D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} + 3(B^{\mu} B_{\mu} - B^{\mu}{}_{|\mu} + \varepsilon D_{\mu}{}^{\mu}{}_{|4}). \tag{10-11}$$

Finally, by using (10-8c) and (10-11) into (10-5c), we obtain (10-9c). ■

Next, we define the *horizontal stress-energy tensor* $T = (T_{\alpha\beta})$ by

$$\begin{aligned}
 T_{\alpha\beta} &= \frac{1}{\bar{k}} \left\{ \bar{k}\bar{T}_{\alpha\beta} - \frac{\varepsilon}{2} E_{\alpha\beta} + \left(\varepsilon D_{\alpha\beta} - \frac{1}{2} F_{\alpha\beta} \right) D_{\mu}{}^{\mu} - \frac{1}{2} F_{\alpha\mu} D_{\beta}{}^{\mu} \right. \\
 &\quad - \frac{1}{2} D_{\alpha\mu} F_{\beta}{}^{\mu} - B_{\alpha|\beta} - \frac{1}{2} F_{\alpha\beta|4} + \varepsilon D_{\alpha\beta|4} + B_{\alpha} B_{\beta} \\
 &\quad \left. - g_{\alpha\beta} \left(B^{\mu} B_{\mu} - B^{\mu}{}_{|\mu} + \frac{\varepsilon}{2} \left((D_{\mu}{}^{\mu})^2 + 2D_{\mu}{}^{\mu}{}_{|4} + D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} \right) \right) \right\}. \tag{10-12}
 \end{aligned}$$

Then, Eq. (10-9a) becomes

$$G_{\alpha\beta} = kT_{\alpha\beta}, \tag{10-13}$$

and are called the *horizontal Einstein equations* on the relativistic general Kaluza-Klein space (\bar{M}, \bar{g}) .

Remark 10.1 Taking into account that all the geometric objects involved in equations (10-9) and (10-13) are either horizontal tensor fields or functions globally defined on \bar{M} , we conclude that these equations are invariant with respect to the four- and cut-transformations (1-7). ■

Corollary 10.1 *The Einstein equations on a relativistic Einstein-Bergmann space (\bar{M}, \bar{g}) are expressed as follows:*

$$\begin{aligned}
 \text{(a)} \quad G_{\alpha\beta} &= \bar{k}\bar{T}_{\alpha\beta} - \frac{\varepsilon}{2} E_{\alpha\beta} + \left(\varepsilon D_{\alpha\beta} - \frac{1}{2} F_{\alpha\beta} \right) D_{\mu}{}^{\mu} - D_{\alpha\mu} F_{\beta}{}^{\mu} \\
 &\quad + \varepsilon D_{\alpha\beta|4} - \frac{\varepsilon}{2} g_{\alpha\beta} \left((D_{\mu}{}^{\mu})^2 + 2D_{\mu}{}^{\mu}|_4 + D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} \right), \\
 \text{(b)} \quad \frac{1}{2} F_{\alpha}{}^{\mu}|_{\mu} &= \varepsilon \left(\bar{k}\bar{T}_{\alpha 4} - D_{\alpha}{}^{\mu}|_{\mu} + D_{\mu}{}^{\mu}|_{\alpha} \right), \\
 \text{(c)} \quad \frac{3}{8} F^{\mu\nu} F_{\mu\nu} &= \bar{k} \left(\bar{T}_{44} - \frac{\varepsilon}{2} \bar{T}_{\alpha\beta} g^{\alpha\beta} \right) + \frac{3}{2} (D_{\mu}{}^{\nu} D_{\nu}{}^{\mu} + D_{\mu}{}^{\mu}|_4). \tag{10.14}
 \end{aligned}$$

Proof. First, we see that (10.14b) and (10.14c) are obtained from (10.9b) and (10.9c) respectively via (5.1). Then, taking into account that in this case $F_{\alpha\beta}$ do not depend on x^4 , and by using (3.27a), we deduce that

$$F_{\alpha\beta|4} = -F_{\alpha\mu} D_{\beta}{}^{\mu} + D_{\alpha\mu} F_{\beta}{}^{\mu}. \tag{10.15}$$

Finally, (10.14a) is obtained from (10.9a), by using (5.1) and (10.15). ■

Remark 10.2 Formally, the horizontal Einstein equations (10.14a) look like equations (35) obtained by Einstein and Bergmann in 7). The main difference is that the Riemannian horizontal connection we use here is a metric connection, while the covariant differentiation used in 7) satisfies only (3.10a) and (3.11a). Also, the adapted frame fields allow us to consider the local components $\bar{T}_{\alpha\beta}$ of the stress-energy tensor \bar{T} . ■

Remark 10.3 Equations (10.14b) have the same form as equations (36) in 7). However, the equations from (10.14c) do not have a correspondent in 7). ■

Next, taking into account that any general Kaluza-Klein space with bundle-like metric is relativistic (see Proposition 7.4), and by using (6.1) and (3.27a), from Theorem 10.1 we deduce the following corollary.

Corollary 10.2 *The Einstein equations on a general Kaluza-Klein space with bundle-like metric are expressed as follows:*

$$\begin{aligned}
 \text{(a)} \quad G_{\alpha\beta} &= \bar{k}\bar{T}_{\alpha\beta} - \frac{\varepsilon}{2} E_{\alpha\beta} - B_{\alpha|\beta} - \frac{1}{2} \frac{\partial F_{\alpha\beta}}{\partial x^4} + B_{\alpha} B_{\beta} - g_{\alpha\beta} (B^{\mu} B_{\mu} - B^{\mu}|_{\mu}), \\
 \text{(b)} \quad \frac{1}{2} F_{\alpha}{}^{\mu}|_{\mu} + B^{\mu} F_{\mu\alpha} &= \varepsilon \bar{k}\bar{T}_{\alpha 4}, \\
 \text{(c)} \quad \frac{3}{8} F^{\mu\nu} F_{\mu\nu} &= \bar{k} \left(\bar{T}_{44} - \frac{\varepsilon}{2} \bar{T}_{\alpha\beta} g^{\alpha\beta} \right) + \frac{3\varepsilon}{2} (B^{\mu} B_{\mu} - B^{\mu}|_{\mu}). \tag{10.16}
 \end{aligned}$$

Finally, taking into account that a classical Kaluza-Klein space is both an Einstein-Bergmann space and a general Kaluza-Klein space with bundle-like metric, from Corollaries 10.1 and 10.2 we deduce the following:

Corollary 10.3 *The Einstein equations on a classical Kaluza-Klein space (\bar{M}, \bar{g}) are expressed as follows:*

$$\begin{aligned}
 \text{(a)} \quad G_{\alpha\beta} &= \bar{k}\bar{T}_{\alpha\beta} - \frac{\varepsilon}{2} E_{\alpha\beta}, & \text{(b)} \quad \frac{1}{2} F_{\alpha}{}^{\mu}|_{\mu} &= \varepsilon \bar{k}\bar{T}_{\alpha 4}, \\
 \text{(c)} \quad \frac{3}{8} F^{\mu\nu} F_{\mu\nu} &= \bar{k} \left(\bar{T}_{44} - \frac{\varepsilon}{2} \bar{T}_{\alpha\beta} g^{\alpha\beta} \right). \tag{10.17}
 \end{aligned}$$

Remark 10.4 From (10-17) we deduce that the local components of the stress-energy tensor \bar{T} with respect to the adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ are necessarily functions of (x^0, x^1, x^2, x^3) alone. ■

§11. A comparison with other approaches

As it is well known, the classical Kaluza-Klein theory was build by using the Levi-Civita connection on both, the spacetime and the 5D space. In the general case, where $g_{\alpha\beta}$ are functions of all five variables, it is raised the question: What is going to be instead of Levi-Civita connection on the spacetime? We answer this question by considering the splitting (1-3) of $T(\bar{M})$, and constructing what we call the Riemannian horizontal connection ∇ on the horizontal distribution $HM\bar{M}$. By using this connection we classify the geodesics of the 5D space (\bar{M}, \bar{g}) , and construct new geometric objects: horizontal Ricci tensor $R_{\alpha\beta}$, horizontal Ricci curvature \mathbf{R} , and horizontal Einstein gravitational tensor field $G_{\alpha\beta}$. It is noteworthy that both $R_{\alpha\beta}$ and $G_{\alpha\beta}$ behave as local components of horizontal tensor fields (or 4D tensors fields), that is, with respect to the transformations (1-7) we have

$$(a) \quad R_{\alpha\beta} = \tilde{R}_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}, \quad (b) \quad G_{\alpha\beta} = \tilde{G}_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}. \quad (11.1)$$

As a consequence of (11.1a) the horizontal scalar curvature \mathbf{R} (see (7-25)) is indeed invariant with respect to (1-7). Moreover, the horizontal Einstein equations (or 4D Einstein equations) given by (10-13) are also invariant with respect to (1-7).

Einstein and Bergmann⁷⁾ defined a 4D covariant differentiation which is only h-metric, that is (3-10a) is satisfied, while (3-10b), in general, is not satisfied. As a consequence of this, there are important differences in the study of curvature tensors of these connections.

Now, we describe the approach of Wesson and his colaborators (see 28)), and compare it with our approach. First, we note that in these papers on space-time-matter theory, the 4D covariant differentiation was given by the local coefficients

$$\Gamma_{\alpha\beta}^{\star\gamma} = \frac{1}{2}g^{\gamma\mu} \left\{ \frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right\}, \quad (11.2)$$

which imitate the Christoffel symbols on the 4D spacetime from classical Kaluza-Klein theory. Performing a long but not a difficult calculation, by taking into account that $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are the local components of horizontal tensor fields of type (0, 2) and (2, 0) respectively, and using (1-8) we deduce that

$$\Gamma_{\alpha\beta}^{\star\gamma} \frac{\partial \tilde{x}^\mu}{\partial x^\gamma} = \tilde{\Gamma}_{\nu\lambda}^{\star\mu} \frac{\partial \tilde{x}^\nu}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\beta} + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta} + \frac{1}{2}\tilde{g}^{\mu\sigma} \left\{ \frac{\partial \tilde{g}_{\sigma\nu}}{\partial \tilde{x}^4} \frac{\partial \tilde{x}^\nu}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} + \frac{\partial \tilde{g}_{\sigma\nu}}{\partial \tilde{x}^4} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \frac{\partial h}{\partial x^\alpha} - \frac{\partial \tilde{g}_{\nu\lambda}}{\partial \tilde{x}^4} \frac{\partial \tilde{x}^\nu}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\beta} \frac{\partial h}{\partial \tilde{x}^\sigma} \right\}, \quad (11.3)$$

with respect to four- and cut-transformations (1-7). Comparing (11.3) with the usual transformations of Christoffel symbols (see (4-9)), we observe that in the general case, were $g_{\alpha\beta}$ do depend on the fifth coordinate, *the local coefficients $\Gamma_{\alpha\beta}^{\star\gamma}$ given by (11.2) do not define a 4D covariant differentiation.* As a direct consequence, the usual Ricci

tensor $R^*_{\alpha\beta}$ constructed by using the local coefficients from (11.2), does not satisfy transformations of the form (11.1a), and therefore the scalar curvature

$$R^* = R^*_{\alpha\beta}g^{\alpha\beta},$$

is not invariant with respect to the transformations (1.7). Thus, the Einstein equations on (\bar{M}, \bar{g}) expressed as in (10.9) do not have tensorial character, and therefore they are not invariant with respect to (1.7). We should remark that if the electromagnetic potentials A_α are not considered in a space-time-matter theory, then the local coefficients from (11.2) coincides with the local coefficients of the Riemannian horizontal connection defined in the present paper (see (3.8)).

Another difference between our approach and the space-time-matter theory developed so far, refers to the 4D electromagnetic tensor field. In the theory we developed in the paper, such a tensor field is defined by (2.4a) and its local components satisfy

$$F_{\alpha\beta} = \tilde{F}_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}, \tag{11.4}$$

with respect to the transformations (1.7). In the space-time-matter theory, the electromagnetic tensor field is defined as in the classical Kaluza-Klein theory, that is, we have (cf. Wesson,²⁸) p. 134)

$$F^*_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}. \tag{11.5}$$

Then by using (1.11) and (1.8a) we obtain

$$F^*_{\alpha\beta} = \tilde{F}^*_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} + \frac{\partial \tilde{A}_\gamma}{\partial \tilde{x}^4} \left(\frac{\partial h}{\partial x^\alpha} \frac{\partial \tilde{x}^\gamma}{\partial x^\beta} - \frac{\partial h}{\partial x^\beta} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \right), \tag{11.6}$$

with respect to (1.7). Comparing (11.6) with (11.4) we see that $F^*_{\alpha\beta}$ do not satisfy tensorial transformations with respect to (1.7). Thus, *any equation which contains $F^*_{\alpha\beta}$ is not invariant with respect to the four- and cut-transformation (1.7)*. Moreover, we cannot apply a 4D covariant differentiation to $F^*_{\alpha\beta}$.

Finally, we note that we consider the general Kaluza-Klein space (\bar{M}, \bar{g}) as a fibre bundle over the 4D space M . In this way we define the induced motions on M as projections of geodesics of (\bar{M}, \bar{g}) on M . Contrary to this situation, in space-time-matter theory, the induced motions an M are defined by using an embedding of M in (\bar{M}, \bar{g}) . Thus, *in this case, the induced motions on M depend on such embedding*. Moreover, if the embedding is locally given by $x^4 = k$, for $k \in R$, then from (1.16) we see that the functions

$$g_{\alpha\beta}(x^\gamma, k) + \epsilon A_\alpha(x^\gamma, k)A_\beta(x^\gamma, k),$$

represent the local components of the induced Lorentz metric on M . Clearly, such a metric is not easy to handle in order to study its curvature. For this reason, most of the papers published on space-time-matter theory, do not consider the electromagnetic potentials into the theory.

§12. Conclusions

We have presented a new point of view on general Kaluza-Klein theories on a $5D$ space. This is based on the Riemannian horizontal connection constructed in §3, which is going to play in this theory, the same role as the Levi-Civita connection on the spacetime, in the classical Kaluza-Klein theory. An important feature of our point of view is the presentation of both the equations of motion and the field equations on the $5D$ space, by using $4D$ geometric objects. This was done by considering on the $5D$ space (\bar{M}, \bar{g}) the adapted frame field $\{\delta/\delta x^\alpha, \partial/\partial x^4\}$ (see (1.10)) and the theory of horizontal tensor fields. For the first time in literature, we split all the equations on (\bar{M}, \bar{g}) into groups of equations expressed in a covariant form, that is, each group of equations is invariant with respect to the four- and cut-transformations (1.7).

We believe that after a correct physical understanding of the fifth dimension (or more general, of the extra dimensions), it will be possible to see practical applications of some general Kaluza-Klein theories. So far, a good part of physics is rather invented than discovered in reality. Hopefully, the new mathematical approach presented in this paper for general Kaluza-Klein theories will be certified by some experimental tests.

Finally, we claim that the present paper can be considered as a starting point for a new approach of non-abelian Kaluza-Klein theories. Our work on this matter is in a good progress, and we hope to finalize it in near future.

References

- 1) A. Bejancu and H. R. Farran, *Foliations and Geometric Structures* (Springer, Dordrecht, 2006).
- 2) A. Billyard and P. S. Wesson, Phys. Rev. D, **53** (1996), 731.
- 3) S. Chatterjee, Gen. Relat. Gravit. **18** (1986), 1073.
- 4) Y. M. Cho, J. Math. Phys. **16** (1975), 2029.
- 5) Y. M. Cho and P. G. O. Freund, Phys. Rev. D **12** (1975), 1711.
- 6) R. Coquereaux and A. Jadczyk, Commun. Math. Phys. **90** (1983), 79.
- 7) A. Einstein and P. Bergmann, Ann. of Math **39** (1938), 683.
- 8) T. Fukui, Gen. Relat. Gravit. **19** (1986), 43.
- 9) U. Günter, A. Starobinsky and A. Zhuk, Phys. Rev. D **69** (2004), 044003.
- 10) T. Kaluza, Sitz. Preuss. Akad. Wiss. Phys. Math. K1 (1921), 966.
- 11) R. Kerner, Ann. Inst. H. Poincaré **9** (1968), 143.
- 12) O. Klein, Z. Phys. **37** (1926), 895.
- 13) S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I (Interscience, New York, 1963).
- 14) T. Liko, J. M. Overduin and P. S. Wesson, Space Sci. Rev. **110** (2004), 337.
- 15) H. Liu and P. S. Wesson, J. Math. Phys. **33** (1992), 3888.
- 16) H. Liu and P. S. Wesson, Gen. Relat. Gravit. **30** (1998), 509.
- 17) B. Mashhoon and P. S. Wesson, Gen. Relat. Gravit. **39** (2007), 1403.
- 18) B. O'Neill, *Semi-Riemannian Geometry and Applications to Relativity* (Academic Press, 1983).
- 19) J. M. Overduin and P. S. Wesson, Phys. Rep. **283** (1997), 303.
- 20) J. Ponce de Leon, Gen. Relat. Gravit. **20** (1988), 539.
- 21) J. Ponce de Leon, Mod. Phys. Lett. A **16** (2001), 1405.
- 22) J. Ponce de Leon and P. S. Wesson, J. Math. Phys. **34** (1993), 4080.
- 23) L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 3370.
- 24) L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 4690.

- 25) B. L. Reinhart, *Ann. of Math.* **69** (1959), 119.
- 26) A. Salam and J. Strathdee, *Ann. of Phys.* **141** (1982), 316.
- 27) S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, New Jersey, 1964).
- 28) P. S. Wesson, *Space-Time-Matter; Modern Kaluza-Klein Theory* (World Scientific, Singapore, 1999).