

# CLASSIFICATION OF 5D WARPED SPACES WITH COSMOLOGICAL CONSTANT

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## Abstract

Let  $(\bar{M}, \bar{g})$  be a 5D warped space defined by the 4D spacetime  $(M, g)$  and the warped function  $A$ . By using the extrinsic curvature of the horizontal distribution, we obtain the classification of all spaces  $(\bar{M}, \bar{g})$  satisfying Einstein equations  $\bar{G} = -\bar{\lambda}\bar{g}$ . This enables us to describe all the exact solutions for the warped metric  $\bar{g}$  by means of 4D exact solutions.

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**Key words:** Cosmological constant; Einstein equations; Exact solutions; Extrinsic curvature; Horizontal Riemannian connection; 5D Warped spaces

## Introduction

There are two versions of  $5D$  relativity known as brane-world theory (cf. Maartens and Koyama [14]) and space-time-matter theory (shortly STM theory) (cf. Overduin and Wesson [18], Wesson [25], P.S. Wesson and J. Ponce de Leon [26]). It seems that both theories share the same concepts and predict the same physics (cf. Ponce de Leon [21]). However, we mark here an important difference between the two theories. Namely, the brane-world theory has by now, a well established model which has been constructed by Randall and Sundrum [22], [23], while in STM theory there are several models (cf. Ponce de Leon [20], Chatterjee and Sil [4], Chatterjee, Panigrahi and Banerjee [5], Liu and Wesson [12], [13], Liu and Mashhoon [11], Mashhoon, Liu and Wesson [15], Billyard and Wesson [3], Liko and Wesson [10], etc.), but so far, no one has had preferential treatment. Moreover, all the models for STM theory are in  $5D$  Ricci flat spaces, while the Randall-Sundrum model has a negative cosmological constant.

The lack of models for STM theory with non-zero cosmological constant is the first motivation of this paper. Then, having in mind the Randall-Sundrum model for brane-world theory, we are motivated to ask for warped models in STM theory. The main purpose of the present paper is to give a classification theorem for  $5D$  warped spaces satisfying Einstein equations with cosmological constant. This is done by using the extrinsic curvature of the horizontal distribution, which actually plays in STM theory the same role as extrinsic curvature in brane-world theory. As a consequence of this general result we find all the exact solutions for Einstein equations in a  $5D$  warped space, four of them being with non-zero cosmological constant. It is worth mentioning that one of these solutions (see (5.36)) has periodic warped local coefficients, which was the main proposal in the first generalization of Kaluza-Klein theory (cf. Einstein and Bergmann [7]).

Now, we outline the content of the paper. In the first section we define the  $5D$  warped space  $(\bar{M}, \bar{g})$  whose metric is given by (1.1). In Section 2 we introduce the extrinsic curvature  $K_{\alpha\beta}$  of the horizontal distribution  $H\bar{M}$ , which is used in Section 3 for the construction of the horizontal Riemannian connection  $\nabla$ . These geometric objects enable us to related in Section 4, the Einstein gravitational tensor fields of  $(\bar{M}, \bar{g})$  and  $(M, g)$ . In Section 5, we find necessary and sufficient conditions for the validity of Einstein equations on  $(\bar{M}, \bar{g})$  (cf. Theorem 5.1) and then state the classification theorem for such spaces (cf. Theorem 5.2). Also, in this section we construct five classes of exact solutions for Einstein equations (5.1). In the last section we discuss

our results in relation with what is known in literature.

## 1 5D Warped Spaces

Let  $(M, g)$  be a spacetime, where  $M$  is a 4-dimensional manifold and  $g$  is a Lorentz metric on  $M$  with signature  $(-+++)$ . Then, consider the product manifold  $\bar{M} = M \times K$ , where  $K$  is a 1-dimensional manifold. The local coordinates on  $\bar{M}$  are  $(x^\alpha, x^4)$ , where  $(x^\alpha)$  and  $(x^4)$  are the local coordinates on  $M$  and  $K$ , respectively. Suppose that  $A$  is a smooth positive function on  $K$ , which is not constant on any coordinate neighbourhood in  $K$ . Also, suppose that  $\bar{g}$  is a semi-Riemannian metric on  $\bar{M}$ , given by its local components:

$$(a) \quad \bar{g}_{\alpha\beta}(x^\alpha) = A(x^4)g_{\alpha\beta}(x^\mu), \quad (b) \quad \bar{g}_{\alpha 4} = 0, \quad (c) \quad \bar{g}_{44} = 1, \quad (1.1)$$

where  $g_{\alpha\beta}(x^\mu)$  are the local components of  $g$ . Hence the interval defined by  $\bar{g}$  has the form

$$dS^2 = A(x^4)g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta + (dx^4)^2. \quad (1.2)$$

Then we call  $(\bar{M}, \bar{g})$  the *5D warped space* defined by the 4D spacetime  $(M, g)$  and the *warp function*  $A$ .

Here, and in the sequel, we use the following ranges for indices:  $\alpha, \beta, \gamma, \dots \in \{0, 1, 2, 3\}$ ;  $a, b, c, \dots \in \{0, 1, 2, 3, 4\}$ . Also, for any vector bundle  $E$  over  $\bar{M}$ , denote by  $\Gamma(E)$  the  $F(\bar{M})$ -module of all smooth sections of  $E$ , where  $F(\bar{M})$  is the algebra of smooth functions on  $\bar{M}$ .

Next, we recall the well known Randall-Sundrum metric given by (cf. [22], [23])

$$dS^2 = e^{-2|x^4|/l} \eta_{\alpha\beta} dx^\alpha dx^\beta + (dx^4)^2. \quad (1.3)$$

where  $l$  is the curvature radius of  $AdS_5$  and  $\eta_{\alpha\beta}$  are the components of the Minkowski metric. Also, we recall that the metric (1.3) is the main concept from the brane-world theory (cf. Maartens and Koyama [14]). Clearly, for each half of the bulk  $x^4 > 0$  or  $x^4 < 0$ , (1.3) is an example of the metric (1.2). For this reason, a metric of the form

$$dS^2 = e^{kx^4} g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \quad (1.4)$$

where  $k$  is a non-zero constant, will be called a *metric of Randall-Sundrum type*.

## 2 The Extrinsic Curvature of the Horizontal Distribution

Let  $H\bar{M}$  and  $V\bar{M}$  be the distributions that are tangent to the foliations with leaves  $M \times \{y\}$  and  $\{x\} \times K$ , respectively. Thus, according to (1.1b), the tangent bundle  $T\bar{M}$  of  $\bar{M}$  has the orthogonal Whitney decomposition

$$T\bar{M} = H\bar{M} \oplus V\bar{M}. \quad (2.1)$$

Denote by the same symbol  $\bar{g}$  the restriction of the semi-Riemannian metric  $\bar{g}$  on  $\bar{M}$  to  $H\bar{M}$ . Then  $(H\bar{M}, \bar{g})$  is a Lorentz distribution on  $\bar{M}$ , while  $V\bar{M}$  is a spacelike distribution. We call  $H\bar{M}$  and  $V\bar{M}$  the *horizontal distribution* and *vertical distribution* on  $\bar{M}$ , respectively.

Next, we define the functions

$$K_{\alpha\beta} = \frac{1}{2} \frac{\partial \bar{g}_{\alpha\beta}}{\partial x^4} = \frac{1}{2} A'(x^4) g_{\alpha\beta}(x^\mu). \quad (2.2)$$

It is easy to check that we have

$$K_{\alpha\beta} = \frac{1}{2} \left( L_{\frac{\partial}{\partial x^4}} \bar{g} \right) \left( \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right), \quad (2.3)$$

where  $L$  is the Lie derivative operator.

Thus  $K_{\alpha\beta}$  are the surviving local components of the extrinsic curvature from brane-world geometry (cf. Maartens and Koyama [14], Sasaki, Shiromizu and Maeda [24]). Also, note that,  $K_{\alpha\beta}(x^a)$  are functions on  $\bar{M}$  which are transformed as the local components of a tensor field of type (0,2) on  $M$ , that is, they define a  $4D$  (adapted) tensor field on  $\bar{M}$  (see Bejancu and Farran [2]). For the above reasons, we call the symmetric  $4D$  tensor field defined by  $K_{\alpha\beta}(x^a)$  on  $\bar{M}$ , the *extrinsic curvature* of the horizontal distribution. Taking into account that the entries of the inverses of the  $4 \times 4$  matrices  $[\bar{g}_{\alpha\beta}]$  and  $[g_{\alpha\beta}]$  are related by

$$\bar{g}^{\alpha\beta} = \frac{1}{A} g^{\alpha\beta}, \quad (2.4)$$

we obtain

$$\begin{aligned} (a) \quad K_\alpha^\gamma &= \bar{g}^{\gamma\beta} K_{\alpha\beta} = \frac{B}{2} \delta_\alpha^\gamma, & (b) \quad K &= K_\alpha^\alpha = 2B, \\ (c) \quad K^{\alpha\beta} &= \bar{g}^{\alpha\gamma} \bar{g}^{\beta\mu} K_{\gamma\mu} = \frac{B}{2A} g^{\alpha\beta}, \end{aligned} \quad (2.5)$$

where we put

$$B = \frac{A'}{A}. \quad (2.6)$$

### 3 The Levi-Civita Connection on $(\bar{M}, \bar{g})$

Denote by  $\bar{\nabla}$  the Levi-Civita connection on  $(\bar{M}, \bar{g})$ , that is,  $\bar{\nabla}$  is given by (cf. O'Neill [17], p.61)

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= X(\bar{g}(Y, Z)) + Y(\bar{g}(Z, X)) - Z(\bar{g}(X, Y)) + \\ &\bar{g}([X, Y], Z) - \bar{g}([Y, Z], X) + \bar{g}([Z, X], Y), \end{aligned} \quad (3.1)$$

for all  $X, Y, Z \in \Gamma(T\bar{M})$ . Then by direct calculations, using (3.1), (1.1), (1.2) and (2.5a) we obtain

$$\begin{aligned} (a) \quad \bar{\nabla}_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\alpha} &= \Gamma_{\alpha \beta}^{\gamma} \frac{\partial}{\partial x^\gamma} - K_{\alpha\beta} \frac{\partial}{\partial x^4}, \quad (b) \quad \bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^\alpha} \\ &= \bar{\nabla}_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^4} = K_{\alpha}^{\gamma} \frac{\partial}{\partial x^\gamma}, \quad (c) \quad \bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4} = 0, \end{aligned} \quad (3.2)$$

where  $\Gamma_{\alpha \beta}^{\gamma}$  are the Christoffel symbols on  $(M, g)$  given by

$$\Gamma_{\alpha \beta}^{\gamma} = \frac{1}{2} g^{\gamma\mu} \left\{ \frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right\}. \quad (3.3)$$

Next, we denote by  $\nabla$  the projection of  $\bar{\nabla}$  on  $H\bar{M}$ , that is, we have

$$(a) \quad \nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\alpha} = \Gamma_{\alpha \beta}^{\gamma} \frac{\partial}{\partial x^\gamma}, \quad (b) \quad \nabla_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^\alpha} = K_{\alpha}^{\gamma} \frac{\partial}{\partial x^\gamma}. \quad (3.4)$$

It is important to note that  $\nabla$  is a metric linear connection on the horizontal Lorentz distribution, that is, we have

$$(a) \quad \bar{g}_{\alpha\beta|\gamma} = 0, \quad (b) \quad \bar{g}_{\alpha\beta|4} = 0, \quad (c) \quad \bar{g}^{\alpha\beta}|_{\gamma} = 0, \quad (d) \quad \bar{g}^{\alpha\gamma}|_4 = 0, \quad (3.5)$$

where " | " stands for covariant derivative with respect to  $\nabla$ . The linear connection  $\nabla$  is known as *Riemannian horizontal connection*, and it is different from the nonholonomic linear connection which has been used in Bejancu [1]. Taking into account (2.2) we deduce that the extrinsic curvature is *horizontally parallel* with respect to  $\nabla$ , that is, we have

$$(a) \quad K_{\alpha\beta|\gamma} = 0, \quad (b) \quad K_{\alpha|\gamma}^{\beta} = 0. \quad (3.6)$$

Also, by using (2.2), (2.5a) and (2.6), we obtain

$$\begin{aligned} (a) \quad K_{\alpha\beta|4} &= \frac{\partial K_{\alpha\beta}}{\partial x^4} - K_{\lambda\beta} K_{\alpha}^{\lambda} - K_{\alpha\lambda} K_{\beta}^{\lambda} = \frac{B'A}{2} g_{\alpha\beta}, \\ (b) \quad K_{\alpha|4}^{\mu} &= \frac{\partial K_{\alpha}^{\mu}}{\partial x^4} = \frac{B'}{2} \delta_{\alpha}^{\mu}. \end{aligned} \quad (3.7)$$

## 4 The Ricci Tensors of $(\bar{M}, \bar{g})$ and $(M, g)$

Let  $\bar{R}$  be the curvature tensor field of  $\bar{\nabla}$  given by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z, \quad (4.1)$$

for all  $X, Y, Z \in \Gamma(T\bar{M})$ . Then, we put

$$\begin{aligned} (a) \quad & h\bar{R}\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\beta}\right) \frac{\partial}{\partial x^\alpha} = \bar{R}_\alpha{}^\mu{}_{\beta\gamma} \frac{\partial}{\partial x^\mu}, \\ (b) \quad & \bar{R}\left(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^\beta}\right) \frac{\partial}{\partial x^\alpha} = \bar{R}_\alpha{}^\mu{}_{\beta 4} \frac{\partial}{\partial x^\mu} + \bar{R}_\alpha{}^4{}_{\beta 4} \frac{\partial}{\partial x^4}, \\ (c) \quad & h\bar{R}\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^4}\right) \frac{\partial}{\partial x^4} = \bar{R}_4{}^\mu{}_{4\alpha} \frac{\partial}{\partial x^\mu}, \end{aligned} \quad (4.2)$$

where  $h$  is the projection morphism of  $T\bar{M}$  on  $H\bar{M}$  with respect to (2.1). Then, by using (4.2), (4.1), (3.2), (3.3) and (3.6b), we deduce that

$$\begin{aligned} (a) \quad & \bar{R}_\alpha{}^\mu{}_{\beta\gamma} = R_\alpha{}^\mu{}_{\beta\gamma} + K_{\alpha\gamma} K_\beta^\mu - K_{\alpha\beta} K_\gamma^\mu, \\ (b) \quad & \bar{R}_\alpha{}^\mu{}_{\beta 4} = 0, \quad (c) \quad \bar{R}_\alpha{}^4{}_{\beta 4} = -K_{\alpha\beta|4} - K_{\alpha\lambda} K_\beta^\lambda, \\ (d) \quad & \bar{R}_4{}^\mu{}_{4\alpha} = -K_{\alpha|4}^\mu - K_\alpha^\lambda K_\lambda^\mu, \end{aligned} \quad (4.3)$$

where  $R_\alpha{}^\mu{}_{\beta\gamma}$  are the local components of the curvature tensor field of the Levi-Civita connection on  $(M, g)$ . Moreover, by using (2.2), (2.5a) and (3.7) in (4.3), we obtain

$$\begin{aligned} (a) \quad & \bar{R}_\alpha{}^\mu{}_{\beta\gamma} = R_\alpha{}^\mu{}_{\beta\gamma} + \frac{1}{4} A' B \left( g_{\alpha\gamma} \delta_\beta^\mu - g_{\alpha\beta} \delta_\gamma^\mu \right), \\ (b) \quad & \bar{R}_\alpha{}^\mu{}_{\beta 4} = 0, \quad (c) \quad \bar{R}_\alpha{}^4{}_{\beta 4} = -\frac{1}{4} (2B'A + A'B) g_{\alpha\beta}, \\ (d) \quad & \bar{R}_4{}^\mu{}_{4\alpha} = -\frac{1}{4} (2B' + B^2) \delta_\alpha^\mu. \end{aligned} \quad (4.4)$$

Next, we consider the Ricci tensor  $\bar{R}ic$  on  $(\bar{M}, \bar{g})$  whose local components

$$\bar{R}_{ab} = \bar{R}ic\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right),$$

are given by

$$\bar{R}_{ab} = \bar{R}_a{}^c{}_{bc}. \quad (4.5)$$

Then, by using (4.5) and (4.4), we obtain

$$\begin{aligned} (a) \quad \bar{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}(2A'B + B'A)g_{\alpha\beta}, & (b) \quad \bar{R}_{\alpha 4} &= 0, \\ (c) \quad \bar{R}_{44} &= -2B' - B^2, \end{aligned} \quad (4.6)$$

where  $R_{\alpha\beta}$  are the local components of the Ricci tensor of the Levi-Civita connection on  $(M, g)$ . As a consequence of (4.6) we deduce that the scalar curvatures  $\bar{\mathbf{R}}$  and  $\mathbf{R}$  of  $(\bar{M}, \bar{g})$  and  $(M, g)$  are related by

$$\bar{\mathbf{R}} = \frac{1}{A}\mathbf{R} - 4B' - 5B^2. \quad (4.7)$$

Finally, denote by  $\bar{G}$  the Einstein gravitational tensor field of  $(\bar{M}, \bar{g})$ , that is, we have

$$\bar{G} = \bar{R}ic - \frac{1}{2}\bar{\mathbf{R}}\bar{g}. \quad (4.8)$$

Then, by using (4.8), (4.7), (4.6) and (1.1), we obtain

$$\begin{aligned} (a) \quad \bar{G}_{\alpha\beta} &= \bar{R}_{\alpha\beta} - \frac{1}{2}\bar{\mathbf{R}}\bar{g}_{\alpha\beta} = G_{\alpha\beta} + \frac{3}{2}A(B' + B^2)g_{\alpha\beta}, \\ (b) \quad \bar{G}_{\alpha 4} &= 0, & (c) \quad \bar{G}_{44} &= \frac{1}{2}(3B^2 - \frac{1}{A}\mathbf{R}), \end{aligned} \quad (4.9)$$

where  $G_{\alpha\beta}$  are the local components of the Einstein gravitational tensor field  $G$  of the spacetime  $(M, g)$ .

## 5 Einstein Field Equations with Cosmological Constant

Let  $(\bar{M}, \bar{g})$  be the  $5D$  warped space defined by the  $4D$  spacetime  $(M, g)$  and the warped function  $A$  (cf. (1.1) and (1.2)). Suppose that the Einstein gravitational tensor field  $\bar{G}$  of  $(\bar{M}, \bar{g})$  satisfies the Einstein equations with cosmological  $\bar{\Lambda}$ , that is, we have

$$\bar{G} = -\bar{\Lambda}\bar{g}. \quad (5.1)$$

First we prove the following.

**Theorem 5.1.** *The Einstein equations (5.1) are equivalent with the following equations*

$$(a) \quad G_{\alpha\beta} = \frac{3}{2}B'Ag_{\alpha\beta}, \quad (b) \quad \bar{\Lambda} = -\frac{3}{2}(2B' + B^2). \quad (5.2)$$

**Proof.** By using (4.9) and (1.1), we deduce that (5.1) is equivalent with the system of equations

$$(a) \quad G_{\alpha\beta} = -A \left\{ \bar{\Lambda} + \frac{3}{2}(B' + B^2) \right\} g_{\alpha\beta}, \quad (5.3)$$

$$(b) \quad \bar{\Lambda} = \frac{1}{2} \left( \frac{1}{A}\mathbf{R} - 3B^2 \right).$$

Then, by contracting (5.3b) with  $g^{\alpha\beta}$  we obtain

$$\frac{1}{A}\mathbf{R} = 4\bar{\Lambda} + 6(B' + B^2). \quad (5.4)$$

Taking into account of (5.4) into (5.3b), we infer that (5.3b) coincides with (5.2b). Finally, using (5.2b) into (5.3a), we deduce that (5.3a) and (5.2a) are equivalent.  $\square$

The form of the equations in (5.2) needs a discussion as follows. First, by contracting (5.2a) with  $g^{\alpha\beta}$  we obtain

$$\mathbf{R} = -6B'A. \quad (5.5)$$

Then (5.2a) is equivalent with

$$R_{\alpha\beta} = -\frac{3}{2}B'Ag_{\alpha\beta}, \quad (5.6)$$

via (5.5). By a well known result (cf. Yano and Kon [27], p.38) we infer that  $B'A$  must be a constant, and therefore  $(M, g)$  is an Einstein manifold. Thus, from Theorem 5.1 we deduce the following.

**Corollary 5.1.** *The Einstein equations (5.1) on  $(\bar{M}, \bar{g})$  with cosmological constant  $\bar{\Lambda}$  induce the Einstein equations*

$$G = -\Lambda g, \quad (5.7)$$

on  $(M, g)$ , where the cosmological constant  $\Lambda$  is given by

$$\Lambda = -\frac{3}{2}B'A. \quad (5.8)$$

Next, we will find all functions  $A$  such that  $\bar{\Lambda}$  given by (5.2b) is a constant. Thus, we must solve the differential equation

$$2B' + B^2 = a, \quad (5.9)$$

where  $a$  is a constant, and then (2.6) for  $A$ . This is done by considering the following four cases.

**Case 1.** Suppose that  $B$  is a non-zero constant function. Thus, we take  $a = 4/L^2$  in (5.9), where  $L$  is a non-zero constant, and from (2.6) we deduce that

$$A = \frac{1}{k}e^{2x^4/L}, \quad k > 0, \quad L \neq 0. \quad (5.10)$$

Note that in this case (5.1) and (5.7) become

$$\bar{G} = \frac{6}{L^2}\bar{g}, \quad (5.11)$$

and

$$G = 0, \quad (5.12)$$

respectively. Thus, the spacetime  $(M, g)$  must be Ricci flat. Moreover, according to (1.4), we conclude that the semi-Riemannian metric  $\bar{g}$  must be a metric of Randall-Sundrum type.

In the other three cases which follow, we suppose that  $B$  is not a constant function.

**Case 2.** ( $a = 0$ ). The solutions of equations (5.9) and (2.6) are given by

$$B = \frac{2}{x^4 + b} \quad \text{and} \quad A = \frac{1}{L^2}(x^4 + b)^2, \quad b \in \mathbb{R}, \quad L \neq 0. \quad (5.13)$$

In this case,  $(\bar{M}, \bar{g})$  is Ricci flat, that is, we have

$$\bar{G} = 0, \quad (5.14)$$

if and only if, on  $(M, g)$  we have the Einstein equations

$$G = -\frac{3}{L^2}g. \quad (5.15)$$

**Case 3.** ( $a = \frac{4}{L^2}$ ,  $L \neq 0$ ). The solutions of equation (5.9) are given by

$$(a) \ B = \frac{2}{L} \tanh \frac{x^4 + b}{L}, \quad \text{and} \quad (b) \ B = \frac{2}{L} \coth \frac{x^4 + b}{L}, \quad b \in R. \quad (5.16)$$

Then, the solutions of (2.6) are given by

$$(a) \ A = \frac{1}{k} \left( \cosh \frac{x^4 + b}{L} \right)^2 \quad \text{and} \quad (b) \ A = \frac{1}{k} \left( \sinh \frac{x^4 + b}{L} \right)^2, \quad (5.17)$$

$b \in R, \quad k > 0.$

Also, in this case on  $(\bar{M}, \bar{g})$  we have the Einstein equations

$$\bar{G} = \frac{6}{L^2} \bar{g}, \quad (5.18)$$

if and only if, on  $(M, g)$  we have the Einstein equations

$$G = \frac{3}{kL^2} g, \quad (5.19)$$

or

$$G = -\frac{3}{kL^2} g. \quad (5.20)$$

**Case 4.** ( $a = -\frac{4}{L^2}$ ,  $L \neq 0$ ). As above, we integrate (5.9) and (2.6), and obtain the solutions

$$B = \frac{2}{L} \tan \frac{b - x^4}{L}, \quad b \in R, \quad (5.21)$$

and

$$A = \frac{1}{k} \left( \cos \frac{b - x^4}{L} \right)^2, \quad b \in R, \quad k > 0, \quad (5.22)$$

respectively. In this case, we deduce that

$$\bar{G} = -\frac{6}{L^2} \bar{g}, \quad \text{on } \bar{M}, \quad (5.23)$$

is equivalent to

$$G = -\frac{3}{kL^2}g, \quad \text{on } M. \quad (5.24)$$

Summing up the above results, we can state the following classification theorem.

**Theorem 5.2.** *Let  $(\bar{M}, \bar{g})$  be a 5D warped space whose Einstein gravitational tensor field satisfies (5.1) with cosmological constant  $\bar{\Lambda}$ . Then  $(\bar{M}, \bar{g})$  must be one of the spaces from the following cases:*

(i) *The cosmological constant of  $(\bar{M}, \bar{g})$  is given by*

$$\bar{\Lambda} = -\frac{6}{L^2}, \quad L \neq 0, \quad (5.25)$$

*and  $\bar{g}$  is given by one of the intervals*

$$\begin{aligned} (a) \quad dS^2 &= \frac{1}{k} e^{2x^4/L} g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \\ (b) \quad dS^2 &= \frac{1}{k} \left( \cosh \frac{b+x^4}{L} \right)^2 g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \\ (c) \quad dS^2 &= \frac{1}{k} \left( \sinh \frac{b+x^4}{L} \right)^2 g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \quad x^4 \neq -b, \end{aligned} \quad (5.26)$$

*where  $k > 0$ , and  $b \in \mathbb{R}$ . Moreover, the spacetime  $(M, g)$  must be Ricci flat in case of the metric (5.26a), and an Einstein space in cases of both metrics (5.26b) and (5.26c), with cosmological constants*

$$\Lambda = -\frac{3}{kL^2}, \quad (5.27)$$

*and*

$$\Lambda = \frac{3}{kL^2}, \quad (5.28)$$

*respectively.*

(ii) *The cosmological constant of  $(\bar{M}, \bar{g})$  and  $\bar{g}$  are given by*

$$\bar{\Lambda} = \frac{6}{L^2}, \quad L \neq 0, \quad (5.29)$$

*and*

$$dS^2 = \frac{1}{k} \left( \cos \frac{b-x^4}{L} \right)^2 g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \quad (5.30)$$

$$x^4 \neq b + L\pi(2h+1)/2, \quad h \in \mathbb{Z},$$

where  $k > 0$ , and  $b \in R$ . In this case,  $(M, g)$  is an Einstein space with cosmological constant

$$\Lambda = \frac{3}{kL^2}. \quad (5.31)$$

(iii) The cosmological constant of  $(\bar{M}, \bar{g})$  is  $\bar{\Lambda} = 0$ , that is,  $(\bar{M}, \bar{g})$  is Ricci flat, and  $\bar{g}$  is given by

$$dS^2 = \frac{1}{L^2}(b + x^4)^2 g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \quad x^4 \neq -b, \quad (5.32)$$

where  $L \neq 0$ , and  $b \in R$ . Moreover,  $(M, g)$  must be an Einstein space with cosmological constant

$$\Lambda = \frac{3}{L^2}. \quad (5.33)$$

By Theorem 5.2 we establish that the spacetime  $(M, g)$  used in the construction of a 5D warped space  $(\bar{M}, \bar{g})$  satisfying the Einstein equations (5.1), must be either Ricci flat for the metric (5.26a), or an Einstein space with cosmological constant  $\Lambda$  given by (5.27), (5.28) or (5.33). This enables us to find exact solutions for  $\bar{g}$  satisfying Einstein equations with cosmological constant  $\bar{\Lambda}$ .

Indeed, if we take  $(M, g)$  as a Minkowski space, then (5.26a) becomes an exact solution for  $\bar{g}$  with cosmological constant given by (5.25). Next, we consider the de Sitter space  $(M, g)$  with cosmological constant  $\Lambda > 0$ , and the interval given by

$$dS^2 = -dt^2 + e^{2(\Lambda/3)^{\frac{1}{2}}t} \{dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\}, \quad (5.34)$$

where  $(r, \theta, \phi)$  are the spherical coordinates. According to Misner, Thorne and Wheeler [14], p.745,  $(M, g)$  with  $g$  given by (5.34) is "the only model satisfying Einstein equations (with  $\Lambda \neq 0$ ) which continually expands and looks the same to any observer who moves with cosmological fluid, regardless of his position or his time". Now, we take  $b = 0$  and  $k = 1$  in (5.26c) and (5.30), and obtain the metrics  $\bar{g}$  given by the intervals

$$dS^2 = \sinh^2\left(\frac{x^4}{L}\right) g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta + (dx^4)^2, \quad x^4 \neq 0, \quad (5.35)$$

and

$$dS^2 = \cos^2\left(\frac{x^4}{L}\right)g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta + (dx^4)^2, \quad x^4 \neq L\pi(2h+1)/2, \quad (5.36)$$

respectively. Note that  $g_{\alpha\beta}(x^\mu)$  in both (5.35) and (5.36), are the local components of the de Sitter metric (5.34) with cosmological constant

$$\Lambda = \frac{3}{L^2}. \quad (5.37)$$

Thus, (5.35) and (5.36) are exact solutions for Einstein equations from (5.1) with cosmological constants,  $\bar{\Lambda} = -6/L^2$  and  $\bar{\Lambda} = 6/L^2$ , respectively. Also, if we take  $b = 0$  in (5.32) we obtain the metric  $\bar{g}$  given by the interval

$$dS^2 = \frac{1}{L^2}(x^4)^2 g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta + (dx^4)^2, \quad x^4 \neq 0, \quad (5.38)$$

where  $g_{\alpha\beta}(x^\mu)$  are the local components of de Sitter metric (5.34) with cosmological constant given by (5.37). Hence (5.38) is an exact solution for Einstein equations from (5.1), with cosmological constant  $\bar{\Lambda} = 0$ . Finally, take  $b = 0$  and  $k = 1$  into (5.26b) and deduce the metric  $\bar{g}$  given by

$$dS^2 = \cosh^2\left(\frac{x^4}{L}\right)g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta + (dx^4)^2, \quad (5.39)$$

where  $g_{\alpha\beta}(x^\mu)$  are the local components of an anti-de Sitter space  $(M, g)$  with cosmological constant  $\Lambda = -3/L^2$ . Thus (5.39) is an exact solution for Einstein equations from (5.1), with cosmological constant  $\bar{\Lambda} = -6/L^2$ .

## 6 Discussions and Conclusions

In the present paper we determine all semi-Riemannian metrics  $\bar{g}$  on a  $5D$  warped space  $\bar{M}$  satisfying the Einstein equations (5.1) (cf. Theorem 5.2). It is worth mentioning that our method is different from what is known in literature. Namely, instead of using the embedding of  $(M, g)$  in  $(\bar{M}, \bar{g})$ , we use the splitting (2.1) of the tangent bundle  $T\bar{M}$  of  $\bar{M}$  and the extrinsic curvature of the horizontal distribution  $H\bar{M}$ . The same method can be applied for any codimension greater than 1.

Now, with respect to the exact solutions for (5.1) presented in the paper, we make the following remarks. Surprisingly, all the solutions found in the

paper have appeared separately in earlier literature, as follows. First, the solution (5.26a) is of Randall-Sundrum type and it was intensively studied with respect to the brane-world theory (cf. Maartens and Koyama [14]). Then the solution (5.38) was first found by Mashhoon, Liu and Wesson [15]. We should note that the metric  $\bar{g}$  in [15] has the signature  $(+ - - - -)$ , but the cosmological constant of  $(M, g)$  has the same value given by (5.37). A particular solution of type (5.38) was considered by Liko and Wesson [10], and also studied by Constantopoulos and Kritikos [6]. We also remark that the solution (5.38) is a particular case of the so called *canonical metric* in a  $5D$  space, which has been intensively studied with respect to the fifth force in a space-time-matter theory (see Chapter 6 in Wesson [25]).

Next, we note that the solutions (5.26a), (5.35) and (5.39) have been used by Emparan, Johnson and Myers [8] in a general study of the equivalence between a gravitational theory in a  $d$ -dimensional anti-de Sitter spacetime and a conformal field theory developed in a  $(d-1)$ -dimensional boundary spacetime. The same solutions (5.26a), (5.35) and (5.39) have also appeared in a study of algebraically special axisymmetric solutions of the higher-dimensional vacuum Einstein equation (cf. Godazgar and Reall [9]). Finally, the solutions (5.35), (5.36) and (5.39) have been used by Park, Pope and Sadrzadeh [19] to obtain brane-world Kaluza-Klein reductions that yield gauge supergravities in the lower dimension from gauged supergravities in the higher dimension.

According to the above remarks, we may claim that all the exact solutions we found for Einstein equations (5.1) have been already taken into consideration in some papers on higher-dimensional physical theories. Further studies on such solutions might reveal new aspects of these theories.

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